THE LARGEST PROPER VARIETY OF LATTICE ORDERED GROUPS

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Abstract. If a lattice ordered group $G$ satisfies any identical relation, other than those satisfied by every lattice ordered group, then $G$ is normal valued, and hence satisfies the relation $ab \leq b^2a^2$ for all $a, b \geq e$.

A lattice ordered group ($l$-group) $G$ is said to be normal valued if whenever $M$ is a convex $l$-subgroup maximal with respect to missing a fixed element $g \in G$, and $K$ the smallest convex $l$-subgroup of $G$ containing $M$ and $g$, then $M$ is a normal subgroup of $K$. It was shown by Wolfenstein [5] that the normal valued $l$-groups are characterized by the property that $ab \leq b^2a^2$ for all $a, b \geq e$, and thus constitute a variety, or equationally defined class (the inequality is equivalent to the equation $|x||y||x^{-2}\vee y^{-2} \vee e = e$, where $|z| = (z \vee z^{-1})$). It has been observed that the variety of normal valued $l$-groups is very large; all of the many varieties studied by Martinez [2] are contained in the normal valued variety (with the exception of the variety of all $l$-groups). It will be shown here that every property variety of lattice ordered groups is contained in the normal valued variety. This sheds a new light on several of Martinez's results, and shows that certain $l$-groups are generic. For example, if $A(R)$ denotes the $l$-group of all order preserving permutations of the real line, and if $A(R)$ satisfies an identical relation, then every $l$-group must satisfy that relation.

Theorem. If a lattice ordered group $G$ satisfies an identical relation which is not satisfied by every lattice ordered group, then $G$ is normal valued.

The Theorem will be proved in a sequence of lemmas.

If an $l$-group $H$ is an $l$-subgroup of the $l$-group $A(S)$ of all order preserving permutations of a totally ordered set $S$, $H$ is said to be o-2-transitive on $S$ if whenever $s_1 < s_2, t_1 < t_2$ are members of $S$, there exists $h \in H$ such that $s_1h = t_1$. It follows easily that any such $H$ must in fact be o-n-transitive in the sense that whenever $s_1 < s_2 < \cdots < s_n, t_1 < t_2 < \cdots < t_n$ are members of $S$, there exists $h \in H$ such that $s_ih = t_i$ [4, Lemma 4]. Again, $H$ is said to be o-primitive on $S$ if $H$ acts transitively on $S$ and the stabilizer subgroups $H_s = \{h \in H | sh = s\}$ are maximal convex $l$-subgroups. Finally, $H$ is periodic on $S$ if $H$ is transitive on $S$ and there is a period $f \in A(S)$, where $S$ is the Dedekind completion of $S$, such that $fh = hf$ for all $h \in H$, where $H \subseteq A(S)$.

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in the natural way, and \( f \) has coterminus orbits. It was shown by McCleary [3] that if \( H \) is \( o \)-primitive on \( S \) then either \( H_s = \{e\} \) for every \( s \in S \), or \( H \) is \( o \)-2-transitive on \( S \), or \( H \) is periodic on \( S \). In the latter case, \( H \) acts faithfully and \( o \)-2-transitively on the interval \((s, sf)\) of \( S \), where \( f \) is the period, and \( s \) is any member of \( S \).

**Lemma 1.** If the \( l \)-subgroup \( H \) of \( A(S) \) is \( o \)-primitive on \( S \) but \( H_s \neq \{e\} \) for some \( s \in S \), then \( H \) contains an \( l \)-subgroup which is \( o \)-2-transitive on some totally ordered set.

Now suppose that \( G \) is an \( l \)-group which is not normal valued. There exists, then, a convex \( l \)-subgroup \( M \) of \( G \), maximal with respect to missing some element \( g \in G \), such that \( M \) is not a normal subgroup of its cover \( K \). The intersection \( \cap k^{-1} Mk \) of all the conjugates of \( M \) in \( K \) is an \( l \)-ideal of \( K \), and the \( l \)-group \( H = K/\cap k^{-1} Mk \) is \( l \)-isomorphic to an \( o \)-primitive \( l \)-subgroup of order preserving permutations of the totally ordered set \( S \) of right cosets of \( M \) in \( K \). In this representation, \( H_s = M \) for some \( s \in S \), and so \( H_s \neq \{e\} \) since \( M \) is not normal in \( K \). By Lemma 1, \( H \) contains an \( l \)-subgroup which is \( o \)-2-transitive on some set. This subgroup must belong to any variety that contains \( G \). Thus:

**Lemma 2.** If \( G \) is an \( l \)-group which is not normal valued, then every variety containing \( G \) contains an \( l \)-group which is an \( o \)-2-transitive \( l \)-group of order preserving permutations of some totally ordered set.

Let \( X \) be a countably infinite set of letters and \( X^{-1} = \{x^{-1} | x \in X \} \) a set disjoint from \( X \). Let \( F \) be the free \( l \)-group on \( X \). The elements of \( F \) may be written in the form \( \forall_A \land_B \Pi_{\Gamma} x_{(\alpha, \beta, \gamma)} \) where \( A, B, \Gamma \) are finite index sets, \( \Gamma = \{1, 2, \ldots, n\} \), \( x_{(\alpha, \beta, \gamma)} \in X \cup X^{-1} \cup \{e\} \), \( \Pi \) indicates the group operation, and \( \lor \) and \( \land \) the lattice operations. There is, in general, nothing unique about the form of a given element of \( F \). An identical relation is, then, a formal expression \( w = e \), where \( w \in F \). An \( l \)-group \( H \) is said to satisfy the identical relation \( w = e \), where \( w \) has the form above, if for every substitution \( x_{(\alpha, \beta, \gamma)} \mapsto h_{(\alpha, \beta, \gamma)} \) by elements of \( H \), we have \( e = \forall_A \land_B \Pi_{\Gamma} h_{(\alpha, \beta, \gamma)} \), where it is understood that if \( h \) is substituted for one occurrence of \( x \), \( h \) must also be substituted for all other occurrences of the same \( x \), \( h^{-1} \) for \( x^{-1} \), and \( e \) (in \( H \)) for \( e \) (in \( F \)), and \( \lor, \land \) indicate the lattice and group operations in \( H \).

**Lemma 3.** Let \( H \) be a nontrivial \( o \)-2-transitive \( l \)-group of order preserving permutations of a totally ordered set \( S \), and \( w \in F \) not the identity element of the free \( l \)-group \( F \). Then \( H \) does not satisfy the identical relation \( w = e \).

To prove Lemma 3, it may first be assumed that \( F \) is an \( l \)-group of order preserving permutations of a totally ordered set \( T \) [1]. There must exist a point \( t \in T \) such that \( tw \neq t \). Let \( w = \forall_A \land_B \Pi_{\Gamma} x_{(\alpha, \beta, \gamma)}, \Gamma = \{1, 2, \ldots, n\} \). For each \( (\alpha, \beta) \in A \times B \), define \( t(\alpha, \beta, 0) = t \), and for \( 1 \leq \gamma \leq n \), \( t(\alpha, \beta, \gamma) = t(\alpha, \beta, \gamma - 1) x_{(\alpha, \beta, \gamma)} \). Now for each \( x \in X \) occurring in \( w \), and each \( (\alpha, \beta) \), let \( P_{(\alpha, \beta)}(x) = \{\gamma | x = x_{(\alpha, \beta, \gamma)} \} \) and \( N_{(\alpha, \beta)}(x) = \{\gamma | x^{-1} = x_{(\alpha, \beta, \gamma)} \} \). Then if \( \gamma \in P_{(\alpha, \beta)}(x), t(\alpha, \beta, \gamma - 1)x = t(\alpha, \beta, \gamma) \), while if \( \gamma \in N_{(\alpha, \beta)}(x), t(\alpha, \beta, \gamma)x = t(\alpha, \beta, \gamma - 1). \)

The set \( T' = \{(\alpha, \beta, \gamma) | (\alpha, \beta) \in A \times B, 0 \leq \gamma \leq n \} \) is a finite subset of \( T \). Now choose and label any subset \( \{s(\alpha, \beta, \gamma) | (\alpha, \beta) \in A \times B, 0 \leq \gamma \leq n \} \) of \( S \).
in one-to-one correspondence with \( T' \), so that the correspondence \( t(\alpha, \beta, \gamma) \leftrightarrow s(\alpha, \beta, \gamma) \) preserves order. Since multiplication by \( x \) provides a one-to-one order preserving correspondence such that

\[
\begin{align*}
  t(\alpha, \beta, \gamma - 1) &\mapsto t(\alpha, \beta, \gamma) \quad \text{for} \ \gamma \in P_{\alpha \beta}(x), \\
  t(\alpha, \beta, \gamma) &\mapsto t(\alpha, \beta, \gamma - 1) \quad \text{for} \ \gamma \in N_{\alpha \beta}(x),
\end{align*}
\]

it follows that the correspondence

\[
\begin{align*}
  s(\alpha, \beta, \gamma - 1) &\mapsto s(\alpha, \beta, \gamma) \quad \text{for} \ \gamma \in P_{\alpha \beta}(x), \\
  s(\alpha, \beta, \gamma) &\mapsto s(\alpha, \beta, \gamma - 1) \quad \text{for} \ \gamma \in N_{\alpha \beta}(x)
\end{align*}
\]

must also be one-to-one and order preserving. As \( H \) is \( 0 \)-transitive on \( S \), \( H \) is also \( 0 \)-transitive, and hence there exists \( h(x) \in H \) such that

\[
\begin{align*}
  s(\alpha, \beta, \gamma - 1)h(x) &\mapsto s(\alpha, \beta, \gamma) \quad \text{for} \ \gamma \in P_{\alpha \beta}(x), \\
  s(\alpha, \beta, \gamma)h(x) &\mapsto s(\alpha, \beta, \gamma - 1) \quad \text{for} \ \gamma \in N_{\alpha \beta}(x).
\end{align*}
\]

Since \( t = t(\alpha, \beta, 0) \) for each \( (\alpha, \beta) \), we may let \( s = s(\alpha, \beta, 0) \). Then substituting \( x \mapsto h(x) \) (and \( x^{-1} \mapsto (h(x))^{-1}, e \mapsto e \)), we have for each \( (\alpha, \beta) \in A \times B \),

\[
s_{\prod \Gamma} h(x_{\alpha \beta}) = s(\alpha, \beta, n).
\]

Since \( tw \neq t \),

\[
t \neq t \land \bigwedge_B \prod \Gamma x_{\alpha \beta} = \bigwedge_A \bigwedge_B \bigwedge \prod \Gamma x_{\alpha \beta} = \bigwedge_A \bigwedge_B t(\alpha, \beta, n),
\]

where on the right side of the equation, the lattice operations are taken on the finite chain \( \{t(\alpha, \beta, n)\} \), which is in one-to-one order preserving correspondence with the chain \( \{s(\alpha, \beta, n)\} \). It follows that

\[
s_{\land A} \land \bigwedge_B \prod \Gamma h(x_{\alpha \beta}) = s_{\land A} \land B \prod \Gamma h(x_{\alpha \beta}) = s_{\land A} \land B s(\alpha, \beta n) \neq s.
\]

Hence \( \land \land B \prod \Gamma h(x_{\alpha \beta}) \neq e \) in \( H \), and \( H \) fails to satisfy the identical relation \( w = e \), proving Lemma 3.

Now to prove the Theorem, let \( G \) be an \( l \)-group which is not normal valued and let \( w = e \) be an identical relation which is not satisfied by every \( l \)-group. By Lemma 2, any variety containing \( G \) must contain an \( l \)-group \( H \) which is \( 0 \)-transitive on some totally ordered set. By Lemma 3, \( H \) fails to satisfy \( w = e \), and hence so does \( G \).

**Corollary.** The variety of normal valued \( l \)-groups is the largest proper variety of \( l \)-groups and contains every other proper variety of \( l \)-groups.

In [2], Martinez showed that the variety \( \mathcal{L} \) of all \( l \)-groups is finitely join irreducible in the lattice of varieties of \( l \)-groups. But a much stronger statement is now obvious.

**Corollary.** The variety \( \mathcal{L} \) of all \( l \)-groups is completely join irreducible in the lattice of varieties of \( l \)-groups.
**Corollary (Martinez [2]).** The variety of normal valued $l$-groups is idempotent.

For a proof, it suffices to produce an $l$-group which is not an extension of a normal valued $l$-group by a normal valued $l$-group. The $l$-group $B(R)$ of all order-preserving permutations of the real line $R$ having bounded support is $l$-simple [1] and not normal valued, and so serves this purpose.

The free $l$-group on a countable set is generic in the sense that it generates the variety of all $l$-groups. A much more tractable example is given by

**Corollary (to Lemma 3).** The $l$-group $A(R)$ of all order-preserving permutations of the real line is generic—if $A(R)$ satisfies a certain identical relation, every $l$-group satisfies that relation.

**References**


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