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THE LARGEST PROPER VARIETY OF LATTICE ORDERED GROUPS

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ABSTRACT. If a lattice ordered group G satisfies any identical relation, other than those satisfied by every lattice ordered group, then G is normal valued, and hence satisfies the relation $ab \leqslant b^2a^2$ for all $a, b \geqslant e$.

A lattice ordered group (l-group) G is said to be *normal valued* if whenever M is a convex l-subgroup maximal with respect to missing a fixed element $g \in G$, and K the smallest convex l-subgroup of G containing M and g, then M is a normal subgroup of K. It was shown by Wolfenstein [5] that the normal valued l-groups are characterized by the property that $ab \leqslant b^2a^2$ for all $a, b \geqslant e$, and thus constitute a *variety*, or equationally defined class (the inequality is equivalent to the equation $|x||y||x|^{-2}|y|^{-2} \vee e = e$, where $|z| = (z \vee z^{-1})$). It has been observed that the variety of normal valued l-groups is very large; all of the many varieties studied by Martinez [2] are contained in the normal valued variety (with the exception of the variety of *all* l-groups). It will be shown here that every property variety of lattice ordered groups is contained in the normal valued variety. This sheds a new light on several of Martinez's results, and shows that certain l-groups are *generic*. For example, if $A(\mathbf{R})$ denotes the l-group of all order preserving permutations of the real line, and if $A(\mathbf{R})$ satisfies an identical relation, then every l-group must satisfy that relation.

THEOREM. *If a lattice ordered group G satisfies an identical relation which is not satisfied by every lattice ordered group, then G is normal valued.*

The Theorem will be proved in a sequence of lemmas.

If an l-group H is an l-subgroup of the l-group $A(S)$ of all order preserving permutations of a totally ordered set S, H is said to be *o-2-transitive* on S if whenever $s_1 < s_2$, $t_1 < t_2$ are members of S, there exists $h \in H$ such that $s_ih = t_i$. It follows easily that any such H must in fact be o-n-transitive in the sense that whenever $s_1 < s_2 < \cdots < s_n$, $t_1 < t_2 < \cdots < t_n$ are members of S, there exists $h \in H$ such that $s_ih = t_i$ [4, Lemma 4]. Again, H is said to be *o-primitive* on S if H acts transitively on S and the stabilizer subgroups $H_s = \{h \in H | sh = s\}$ are maximal convex l-subgroups. Finally, H is *periodic* on S if H is transitive on S and there is a period $f \in A(\overline{S})$, where $\overline{S}$ is the Dedekind completion of S, such that $fh = hf$ for all $h \in H$, where $H \subseteq A(\overline{S})$

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in the natural way, and f has coterminal orbits. It was shown by McCleary [3] that if H is o-primitive on S then either $H_s = \{e\}$ for every $s \in S$, or H is o-2-transitive on S, or H is periodic on S. In the latter case, H_s acts faithfully and o-2-transitively on the interval (s, sf) of S, where f is the period, and s is any member of S.

LEMMA 1. *If the l-subgroup H of $A(S)$ is o-primitive on S but $H_s \neq \{e\}$ for some $s \in S$, then H contains an l-subgroup which is o-2-transitive on some totally ordered set.*

Now suppose that G is an l-group which is not normal valued. There exists, then, a convex l-subgroup M of G, maximal with respect to missing some element $g \in G$, such that M is not a normal subgroup of its cover K. The intersection $\bigcap k^{-1}Mk$ of all the conjugates of M in K is an l-ideal of K, and the l-group $H = K/\bigcap k^{-1}Mk$ is l-isomorphic to an o-primitive l-subgroup of order preserving permutations of the totally ordered set S of right cosets of M in K. In this representation, $H_s = M$ for some $s \in S$, and so $H_s \neq \{e\}$ since M is not normal in K. By Lemma 1, H contains an l-subgroup which is o-2-transitive on some set. This subgroup must belong to any variety that contains G. Thus:

LEMMA 2. *If G is an l-group which is not normal valued, then every variety containing G contains an l-group which is an o-2-transitive l-group of order preserving permutations of some totally ordered set.*

Let X be a countably infinite set of letters and $X^{-1} = \{x^{-1} | x \in X\}$ a set disjoint from X. Let F be the free l-group on X. The elements of F may be written in the form $\bigvee_A \bigwedge_B \prod_\Gamma x_{\alpha\beta\gamma}$ where A, B, Γ are finite index sets, $\Gamma = \{1, 2, \ldots, n\}$, $x_{\alpha\beta\gamma} \in X \cup X^{-1} \cup \{e\}$, $\prod$ indicates the group operation, and $\bigvee$ and $\bigwedge$ the lattice operations. There is, in general, nothing unique about the form of a given element of F. An identical relation is, then, a formal expression $w = e$, where $w \in F$. An l-group H is said to satisfy the identical relation $w = e$, where w has the form above, if for every substitution $x_{\alpha\beta\gamma} \mapsto h_{\alpha\beta\gamma}$ by elements of H, we have $e = \bigvee_A \bigwedge_B \prod_\Gamma h_{\alpha\beta\gamma}$, where it is understood that if h is substituted for one occurrence of x, h must also be substituted for all other occurrences of the same x, h^{-1} for x^{-1}, and e (in H) for e (in F), and $\bigvee$, $\bigwedge$, $\prod$ indicate the lattice and group operations in H.

LEMMA 3. *Let H be a nontrivial o-2-transitive l-group of order preserving permutations of a totally ordered set S, and $w \in F$ not the identity element of the free l-group F. Then H does not satisfy the identical relation $w = e$.*

To prove Lemma 3, it may first be assumed that F is an l-group of order preserving permutations of a totally ordered set T [1]. There must exist a point $t \in T$ such that $tw \neq t$. Let $w = \bigvee_A \bigwedge_B \prod_\Gamma x_{\alpha\beta\gamma}$, $\Gamma = \{1, 2, \ldots, n\}$. For each $(\alpha, \beta) \in A \times B$, define $t(\alpha, \beta, 0) = t$, and for $1 \leqslant \gamma \leqslant n$, $t(\alpha, \beta, \gamma) = t(\alpha, \beta, \gamma - 1)x_{\alpha\beta\gamma}$. Now for each $x \in X$ occurring in w, and each (α, β), let $P_{\alpha\beta}(x) = \{\gamma | x = x_{\alpha\beta\gamma}\}$ and $N_{\alpha\beta}(x) = \{\gamma | x^{-1} = x_{\alpha\beta\gamma}\}$. Then if $\gamma \in P_{\alpha\beta}(x)$, $t(\alpha, \beta, \gamma - 1)x = t(\alpha, \beta, \gamma)$, while if $\gamma \in N_{\alpha\beta}(x)$, $t(\alpha, \beta, \gamma)x = t(\alpha, \beta, \gamma - 1)$.

The set $T' = \{t(\alpha, \beta, \gamma) | (\alpha, \beta) \in A \times B, 0 \leqslant \gamma \leqslant n\}$ is a finite subset of T. Now choose and label any subset $\{s(\alpha, \beta, \gamma) | (\alpha, \beta) \in A \times B, 0 \leqslant \gamma \leqslant n\}$ of S

in one-to-one correspondence with T', so that the correspondence $t(\alpha,\beta,\gamma) \leftrightarrow s(\alpha,\beta,\gamma)$ preserves order. Since multiplication by x provides a one-to-one order preserving correspondence such that

$$t(\alpha,\beta,\gamma-1) \mapsto t(\alpha,\beta,\gamma) \quad \text{for } \gamma \in P_{\alpha\beta}(x),$$
$$t(\alpha,\beta,\gamma) \mapsto t(\alpha,\beta,\gamma-1) \quad \text{for } \gamma \in N_{\alpha\beta}(x),$$

it follows that the correspondence

$$s(\alpha,\beta,\gamma-1) \mapsto s(\alpha,\beta,\gamma) \quad \text{for } \gamma \in P_{\alpha\beta}(x),$$
$$s(\alpha,\beta,\gamma) \mapsto s(\alpha,\beta,\gamma-1) \text{ for } \gamma \in N_{\alpha\beta}(x)$$

must also be one-to-one and order preserving. As H is o-2-transitive on S, H is also o-n-transitive, and hence there exists $h(x) \in H$ such that

$$s(\alpha,\beta,\gamma-1)h(x) = s(\alpha,\beta,\gamma) \quad \text{for } \gamma \in P_{\alpha\beta}(x),$$
$$s(\alpha,\beta,\gamma)h(x) = s(\alpha,\beta,\gamma-1) \text{ for } \gamma \in N_{\alpha\beta}(x).$$

Since $t = t(\alpha,\beta,0)$ for each (α,β), we may let $s = s(\alpha,\beta,0)$. Then substituting $x \mapsto h(x)$ (and $x^{-1} \mapsto h(x^{-1}) = (h(x))^{-1}$, $e \mapsto e$), we have for each $(\alpha,\beta) \in A \times B$,

$$s \textstyle\prod_\Gamma h(x_{\alpha\beta\gamma}) = s(\alpha,\beta,n).$$

Since $tw \neq t$,

$$t \neq t \textstyle\bigvee_A \bigwedge_B \prod_\Gamma x_{\alpha\beta\gamma} = \bigvee_A \bigwedge_B t \prod_\Gamma x_{\alpha\beta\gamma} = \bigvee_A \bigwedge_B t(\alpha,\beta,n),$$

where on the right side of the equation, the lattice operations are taken on the finite chain $\{t(\alpha,\beta,n)\}$, which is in one-to-one order preserving correspondence with the chain $\{s(\alpha,\beta,n)\}$. It follows that

$$s \textstyle\bigvee_A \bigwedge_B \prod_\Gamma h(x_{\alpha\beta\gamma}) = \bigvee_A \bigwedge_B s \prod_\Gamma h(x_{\alpha\beta\gamma}) = \bigvee_A \bigwedge_B s(\alpha,\beta n) \neq s.$$

Hence $\bigvee_A \bigwedge_B \prod_\Gamma h(x_{\alpha\beta\gamma}) \neq e$ in H, and H fails to satisfy the identical relation $w = e$, proving Lemma 3.

Now to prove the Theorem, let G be an l-group which is not normal valued and let $w = e$ be an identical relation which is not satisfied by every l-group. By Lemma 2, any variety containing G must contain an l-group H which is o-2-transitive on some totally ordered set. By Lemma 3, H fails to satisfy $w = e$, and hence so does G.

COROLLARY. *The variety of normal valued l-groups is the largest proper variety of l-groups and contains every other proper variety of l-groups.*

In [2], Martinez showed that the variety $\mathfrak{L}$ of all l-groups is finitely join irreducible in the lattice of varieties of l-groups. But a much stronger statement is now obvious.

COROLLARY. *The variety $\mathfrak{L}$ of all l-groups is completely join irreducible in the lattice of varieties of l-groups.*

COROLLARY (MARTINEZ [2]). *The variety of normal valued l-groups is idempotent.*

For a proof, it suffices to produce an l-group which is not an extension of a normal valued l-group by a normal valued l-group. The l-group $B(\mathbf{R})$ of all order-preserving permutations of the real line $\mathbf{R}$ having bounded support is l-simple [1] and not normal valued, and so serves this purpose.

The free l-group on a countable set is *generic* in the sense that it generates the variety of all l-groups. A much more tractable example is given by

COROLLARY (TO LEMMA 3). *The l-group $A(\mathbf{R})$ of all order-preserving permutations of the real line is generic–if $A(\mathbf{R})$ satisfies a certain identical relation, every l-group satisfies that relation.*

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