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THE SIGNATURE AND ARITHMETIC GENUS OF CERTAIN ASPHERICAL MANIFOLDS

F. THOMAS FARRELL[1]

ABSTRACT. In this paper we show that the signature and arithmetic genus of certain aspherical manifolds M vanish when the center of $\pi_1 M$ is nontrivial. We make the possibly technical assumption that $\pi_1 M$ is residually finite.

0. **Introduction.** Gottlieb [4] has shown that the Euler characteristic of a finite, aspherical polyhedron X vanishes, provided $\pi_1 X$ has a nontrivial center. (Recall that X is aspherical, if $\pi_i X$ vanishes for all $i \neq 1$.)

In this note, we show that the signature of a closed, smooth, aspherical manifold M vanishes, provided $\pi_1 M$ has a nontrivial center and is residually finite. (A group Γ is residually finite if, for each $\gamma \in \Gamma$, there exists a subgroup Δ_γ with finite index such that $\gamma \notin \Delta_\gamma$.) We also prove an analogous result for the arithmetic genus (and the generalized arithmetic genus) of a Kaehler manifold.

The residually finite condition is possibly superfluous, but the author so far has been unable to remove it. (Stallings' paper [7] may be helpful here.) In any event, Malcev [6, Theorem VII] has given the following useful criterion for residual finiteness. Namely Γ is residually finite if, for each $\gamma \in \Gamma$, there exists a representation $\varphi\colon \Gamma \to \mathrm{GL}\,(n, \mathbf{R})$ (where n can vary with γ) such that $\varphi(\gamma)$ is not the identity matrix.

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1. **The signature.** We begin by paraphrasing a basic result from [4].

COROLLARY [4, I.14]. *Let X be a finite, aspherical complex and α an element in the center of $\pi_1(X, x_0)$; then there exists a homotopy $h\colon X \times [0,1] \to X$ such that*

1°. *$h(x,0) = h(x,1) = x$ for all $x \in X$, and*

2°. *the closed path $h(x_0, t)$ represents α.*

This has the following immediate consequence.

LEMMA 1.1. *Let X be a finite, aspherical complex such that $\pi_1 X$ is residually finite and contains a nontrivial center; then there exists a connected, finite-sheeted,*

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regular covering space f: $\overline{X} \to X$ and a covering transformation T: $\overline{X} \to \overline{X}$ such that T is homotopic to the identity map but different from it.

PROOF. Let α be a nontrivial element in the center of $\pi_1(X, x_0)$, h the homotopy posited in Corollary I.14 of [4], and Γ a normal subgroup of $\pi_1(X, x_0)$ with finite index such that $\alpha \notin \Gamma$. Then f: $\overline{X} \to X$ is the regular covering space corresponding to Γ and T is obtained as follows. Lift h to a homotopy $\bar{h}$: $\overline{X} \times [0,1] \to \overline{X}$ such that $\bar{h}(x, 0) = x$ for all $x \in \overline{X}$, and define T by the equation $T(x) = \bar{h}(x, 1)$.

Let M^{4n} be a connected, closed, smooth, oriented manifold of dimension $4n$, then the cup-product pairing evaluated on the orientation class of M defines a symmetric, nondegenerate, bilinear form B on $H^{2n}(M, \mathbf{R})$. Split $H^{2n}(M, \mathbf{R})$ as the direct sum of two subspaces $H^{2n}(M, \mathbf{R}) = H^+ \oplus H^-$ so that B is positive definition on H^+ and negative definite on H^-, and recall that the signature of M, Sign(M), is defined by

$$\text{Sign}\,(M) = \dim H^+ - \dim H^-.$$

In addition, if T: $M \to M$ generates a finite group of orientation preserving diffeomorphisms, then H^+ and H^- can be constructed to be invariant under the automorphism T^* of $H^{2n}(M, \mathbf{R})$ induced by T. In this case, Sign (T, M) is defined by

$$\text{Sign}\,(T, M) = \text{Trace}\,(T^*/H^+) - \text{Trace}\,(T^*/H^-).$$

THEOREM 1.2. *Let M^{4n} be a closed, smooth, aspherical manifold, then* Sign(M) *vanishes provided $\pi_1 M$ is residually finite and contains a nontrivial center.*

PROOF. Let f: $\overline{M} \to M$ and T: $\overline{M} \to \overline{M}$ be the covering space and transformation posited in 1.1, and m denote the number of sheets of f. Since f is a degree m, codimension-0 immersion, the Hirzebruch signature theorem [5, Theorem 8.2.2] implies that Sign $(\overline{M}) = m$ Sign (M). Hence it suffices to show that Sign $(\overline{M})$ vanishes. By 1.1, T^* is the identity map, therefore Sign $(\overline{M})$ = Sign $(T, \overline{M})$. But T has no fixed points, so an application of the Atiyah-Singer G-signature theorem [2, Theorem 6.12] yields that Sign $(T, \overline{M})$ vanishes.

2. **The arithmetic genus.** We start by recalling the definition of the arithmetic genus. Let M^n be a complex analytic manifold of complex dimension n, and A^k denote the $\mathbf{C}$-module of global smooth $\mathbf{C}$-valued k-forms on M. Then A^k splits as the direct sum $A^k = \sum_{p+q=k} A^{p,q}$ where $A^{p,q}$ denotes the $\mathbf{C}$-module of global forms of type (p, q) on M. And the exterior derivative d decomposes as $d = \partial + \bar{\partial}$, where ∂: $A^{p,q} \to A^{p+1,q}$ and $\bar{\partial}$: $A^{p,q} \to A^{p,q+1}$. (Here ∂ and $\bar{\partial}$ are differentiation with regard to the z-variable and $\bar{z}$-variable respectively.) Now for each p,

$$A^{p,0} \xrightarrow{\bar{\partial}} A^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} A^{p,n}$$

is a cochain complex whose cohomology we denote by $H^{p,q}$,and the arithmetic genus $\chi(M)$ is given by

$$\chi(M) = \sum_{q=0}^{n} (-1)^q \dim H^{0,q}.$$

More generally $\chi^p(M)$ is defined by

$$\chi^p(M) = \sum_{q=0}^{n} (-1)^q \dim H^{p,q},$$

and the polynomial $\chi_y(M)$, called the generalized arithmetic genus, by

$$\chi_y(M) = \sum_{p=0}^{n} \chi^p(M) y^p.$$

THEOREM 2.1. *Let M^n be an aspherical, closed Kaehler manifold, then $\chi_y(M)$ vanishes provided $\pi_1 M$ is residually finite and contains a nontrivial center.*

REMARK 2.2. Borel and Hirzebruch have obtained some pertinent calculations of $\chi_y(M)$ (hence also Sign(M)) for an important class of aspherical, Kaehler manifolds. (See [5, §22.2] and [3].)

PROOF. Let f: $\overline{M} \to M$ and T: $\overline{M} \to \overline{M}$ be the covering space and transformation given by 1.1; then the Riemann-Roch theorem [2, Theorem 4.3] implies that $\chi^p(\overline{M}) = m\chi^p(M)$ where m is the number of sheets of f. Hence it suffices to show that $\chi^p(\overline{M})$ vanishes for $p = 0, 1, \ldots, n$.

Now the complex analytic map T induces endomorphisms $T^{p,q}$ of $A^{p,q}$ which commute with $\bar{\partial}$ (as well as d). Denote the induced endomorphism of $H^{p,q}$ by $H^{p,q}(T)$, and define the pth Lefschetz number $L^p(T)$ by the formula

$$L^p(T) = \sum_{q=0}^{n} (-1)^q \operatorname{trace} H^{p,q}(T).$$

Since T has no fixed points, the holomorphic form of the Atiyah-Bott fixed point theorem [1, formula 4.9] implies that $L^p(T) = 0$. Therefore it remains to show that $\chi^p(\overline{M}) = L^p(T)$, which is a result of the following.

LEMMA 2.3. *For all p and q, $H^{p,q}(T)$ is the identity endomorphism.*

PROOF. Use the Kaehler metric on $\overline{M}$ induced by f to define the complex Laplace-Beltrami operator $\square$: $A^{p,q} \to A^{p,q}$ and the real Laplace operator Δ: $A^k \to A^k$; then $\Delta = 2\,\square$. (See [5, pp. 121–124] for more details.) Because T is an isometry, $T^{p,q}$ commutes with Δ, thus also with $\square$, and therefore leaves the kernel of $\square$ invariant. Since $\ker \bar{\partial} = \ker \square \oplus \operatorname{image} \bar{\partial}$, it suffices to show that $T^{p,q}$ restricted to $\ker \square$ is the identity map. But this is a consequence of the Poincaré lemma, which says that homotopic maps (T and the identity) induce chain homotopic maps on the De Rham complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \xrightarrow{d} A^{2n},$$

together with the equations

$$\ker d = \ker \Delta \oplus \operatorname{image} d \quad \text{and} \quad \ker \Delta = \ker \square\,.$$

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Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802