THE SIGNATURE AND ARITHMETIC GENUS
OF CERTAIN ASPHERICAL MANIFOLDS

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Abstract. In this paper we show that the signature and arithmetic genus of certain aspherical manifolds \( M \) vanish when the center of \( \pi_1 M \) is nontrivial. We make the possibly technical assumption that \( \pi_1 M \) is residually finite.

0. Introduction. Gottlieb [4] has shown that the Euler characteristic of a finite, aspherical polyhedron \( X \) vanishes, provided \( \pi_1 X \) has a nontrivial center. (Recall that \( X \) is aspherical, if \( \pi_i X \) vanishes for all \( i \neq 1 \).)

In this note, we show that the signature of a closed, smooth, aspherical manifold \( M \) vanishes, provided \( \pi_1 M \) has a nontrivial center and is residually finite. (A group \( \Gamma \) is residually finite if, for each \( \gamma \in \Gamma \), there exists a subgroup \( \Delta_\gamma \) with finite index such that \( \gamma \not\in \Delta_\gamma \).) We also prove an analogous result for the arithmetic genus (and the generalized arithmetic genus) of a Kaehler manifold.

The residually finite condition is possibly superfluous, but the author so far has been unable to remove it. (Stallings' paper [7] may be helpful here.) In any event, Malcev [6, Theorem VII] has given the following useful criterion for residual finiteness. Namely \( \Gamma \) is residually finite if, for each \( \gamma \in \Gamma \), there exists a representation \( \varphi: \Gamma \to \text{GL}(n, \mathbb{R}) \) (where \( n \) can vary with \( \gamma \)) such that \( \varphi(\gamma) \) is not the identity matrix.

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1. The signature. We begin by paraphrasing a basic result from [4].

Corollary [4, I.14]. Let \( X \) be a finite, aspherical complex and \( \alpha \) an element in the center of \( \pi_1(X, x_0) \); then there exists a homotopy \( h: X \times [0,1] \to X \) such that

1°. \( h(x, 0) = h(x, 1) = x \) for all \( x \in X \), and

2°. the closed path \( h(x_0, t) \) represents \( \alpha \).

This has the following immediate consequence.

Lemma 1.1. Let \( X \) be a finite, aspherical complex such that \( \pi_1 X \) is residually finite and contains a nontrivial center; then there exists a connected, finite-sheeted,
regular covering space \( f: \overline{X} \to X \) and a covering transformation \( T: \overline{X} \to \overline{X} \) such that \( T \) is homotopic to the identity map but different from it.

**Proof.** Let \( \alpha \) be a nontrivial element in the center of \( \pi_1(X, x_0) \), \( h \) the homotopy posited in Corollary 1.14 of [4], and \( \Gamma \) a normal subgroup of \( \pi_1(X, x_0) \) with finite index such that \( \alpha \not\in \Gamma \). Then \( f: \overline{X} \to X \) is the regular covering space corresponding to \( \Gamma \) and \( T \) is obtained as follows. Lift \( h \) to a homotopy \( \tilde{h}: \overline{X} \times [0,1] \to \overline{X} \) such that \( \tilde{h}(x,0) = x \) for all \( x \in \overline{X} \), and define \( T \) by the equation \( T(x) = \tilde{h}(x,1) \).

Let \( M^{4n} \) be a connected, closed, smooth, oriented manifold of dimension \( 4n \), then the cup-product pairing evaluated on the orientation class of \( M \) defines a symmetric, nondegenerate, bilinear form \( B \) on \( H^{2n}(M, \mathbb{R}) \). Split \( H^{2n}(M, \mathbb{R}) \) as the direct sum of two subspaces \( H^{2n}(M, \mathbb{R}) = H^+ \oplus H^- \) so that \( B \) is positive definite on \( H^+ \) and negative definite on \( H^- \), and recall that the signature of \( M \), \( \text{Sign}(M) \), is defined by

\[
\text{Sign}(M) = \dim H^+ - \dim H^-.
\]

In addition, if \( T: M \to M \) generates a finite group of orientation preserving diffeomorphisms, then \( H^+ \) and \( H^- \) can be constructed to be invariant under the automorphism \( T^* \) of \( H^{2n}(M, \mathbb{R}) \) induced by \( T \). In this case, \( \text{Sign}(T,M) \) is defined by

\[
\text{Sign}(T,M) = \text{Trace}(T^*/H^+ - \text{Trace}(T^*/H^-).
\]

**Theorem 1.2.** Let \( M^{4n} \) be a closed, smooth, aspherical manifold, then \( \text{Sign}(M) \) vanishes provided \( \pi_1(M) \) is residually finite and contains a nontrivial center.

**Proof.** Let \( f: M \to M \) and \( T: M \to M \) be the covering space and transformation posited in 1.1, and \( m \) denote the number of sheets of \( f \). Since \( f \) is a degree \( m \), codimension-0 immersion, the Hirzebruch signature theorem [5, Theorem 8.2.2] implies that \( \text{Sign}(M) = m \text{Sign}(M) \). Hence it suffices to show that \( \text{Sign}(M) \) vanishes. By 1.1, \( T^* \) is the identity map, therefore \( \text{Sign}(M) = \text{Sign}(T,M) \). But \( T \) has no fixed points, so an application of the Atiyah-Singer G-signature theorem [2, Theorem 6.12] yields that \( \text{Sign}(T,M) \) vanishes.

**2. The arithmetic genus.** We start by recalling the definition of the arithmetic genus. Let \( M^n \) be a complex analytic manifold of complex dimension \( n \), and \( \mathbb{A}^k \) denote the \( \mathbb{C} \)-module of global smooth \( \mathbb{C} \)-valued \( k \)-forms on \( M \). Then \( \mathbb{A}^k \) splits as the direct sum \( \mathbb{A}^k = \sum_{p+q=k} \mathbb{A}^{p,q} \) where \( \mathbb{A}^{p,q} \) denotes the \( \mathbb{C} \)-module of global forms of type \( (p,q) \) on \( M \). And the exterior derivative \( d \) decomposes as \( d = \partial + \overline{\partial} \), where \( \partial: \mathbb{A}^{p,q} \to \mathbb{A}^{p+1,q} \) and \( \overline{\partial}: \mathbb{A}^{p,q} \to \mathbb{A}^{p,q+1} \). (Here \( \partial \) and \( \overline{\partial} \) are differentiation with regard to the \( z \)-variable and \( \overline{z} \)-variable respectively.) Now for each \( p \),

\[
\mathbb{A}^{p,0} \xrightarrow{\overline{\partial}} \mathbb{A}^{p,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathbb{A}^{p,n}
\]

is a cochain complex whose cohomology we denote by \( H^{p,q} \), and the arithmetic genus \( \chi(M) \) is given by
Theorem 2.1. Let $M^n$ be an aspherical, closed Kaehler manifold, then $\chi_y(M)$ vanishes provided $\pi_1 M$ is residually finite and contains a nontrivial center.

Remark 2.2. Borel and Hirzebruch have obtained some pertinent calculations of $\chi_y(M)$ (hence also $\text{Sign}(M)$) for an important class of aspherical, Kaehler manifolds. (See [5, §22.2] and [3].)

Proof. Let $f: M \to M$ and $T: M \to M$ be the covering space and transformation given by 1.1; then the Riemann-Roch theorem [2, Theorem 4.3] implies that $\chi^p(M) = n \chi_x^p(M)$ where $n$ is the number of sheets of $f$. Hence it suffices to show that $\chi^p(M)$ vanishes for $p = 0, 1, \ldots, n$. Now the complex analytic map $T$ induces endomorphisms $T_{p,q}$ of $A_{p,q}$ which commute with $\bar{\partial}$ (as well as $\partial$). Denote the induced endomorphism of $H_{p,q}$ by $H_{p,q}(T)$, and define the $p$th Lefschetz number $L_p(T)$ by the formula

$$L_p(T) = \sum_{q=0}^{n} (-1)^q \text{trace } H_{p,q}(T).$$

Since $T$ has no fixed points, the holomorphic form of the Atiyah-Bott fixed point theorem [1, formula 4.9] implies that $L_p(T) = 0$. Therefore it remains to show that $\chi^p(M) = L_p(T)$, which is a result of the following.

Lemma 2.3. For all $p$ and $q$, $H_{p,q}(T)$ is the identity endomorphism.

Proof. Use the Kaehler metric on $\overline{M}$ induced by $f$ to define the complex Laplace-Beltrami operator $\Box: A_{p,q} \to A_{p,q}$ and the real Laplace operator $\Delta: A^k \to A^k$; then $\Delta = 2 \Box$. (See [5, pp. 121–124] for more details.) Because $T$ is an isometry, $T_{p,q}$ commutes with $\Delta$, thus also with $\Box$, and therefore leaves the kernel of $\Box$ invariant. Since $\ker \bar{\partial} = \ker \Box \oplus \text{image } \delta$, it suffices to show that $T_{p,q}$ restricted to $\ker \Box$ is the identity map. But this is a consequence of the Poincaré lemma, which says that homotopic maps ($T$ and the identity) induce chain homotopic maps on the De Rham complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} \cdots \xrightarrow{d} A^{2n},$$

together with the equations

$$\ker d = \ker \Delta \oplus \text{image } d \quad \text{and} \quad \ker \Delta = \ker \Box.$$
REFERENCES


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