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LOCALLY COMPACT SUBGROUPS OF METRIZABLE TOPOLOGICAL ABELIAN GROUPS

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ABSTRACT. This paper examines the question of existence of locally compact subgroups of topological abelian groups. A counterexample is given and it is shown in a certain setting that groups have a nontrivial locally compact subgroup.

1. **Introduction.** If H is an infinite, locally compact topological abelian group, then H has a nontrivial, proper locally compact subgroup H_1. More generally, suppose H is an infinite, complete, metrizable, topological abelian group. We will see that H may not have any nontrivial, locally compact subgroups.

When such a topological group H is a subgroup of a quotient group of the Banach space C_0 modulo the integer valued sequences, H will have a nontrivial locally compact subgroup. This will follow from a more general result involving Banach space with bases. The results in this paper are established by constructions utilizing coordinates.

2. **A group with no nontrivial locally compact subgroups.** The following construction will yield an example of an infinite, complete, metrizable topological group H whose only locally compact subgroup is the trivial one.

Let $\{u_n\}_{n=1}^{\infty}$ be the standard basis for l_1, $v_n = (1/n)u_n$, and K be the closed subgroup of l_1 generated by $\{v_n\}_{n=1}^{\infty}$. Thus if $x \in K$, we can write x in standard l_1 coordinates as $x = (p_1/1, p_2/2, p_3/3, \dots)$ with $\sum_{n=1}^{\infty} |p_n|/n < \infty$ and the p_n integers. If $y \in l_1$ with $y = (a_1, a_2, \dots)$, we can write $y = x + z$ with $x \in K$, $z \in l_1$, $z = (\beta_1, \beta_2, \dots)$ and $-1/2n < \beta_n \leqslant 1/2n$ for each n.

Define L to be the quotient group $L = l_1/K$. If $w \in L$, w is the image under the quotient map of some $y \in l_1$ with y as above, $y = x + z$, etc. We will represent w as $w = (\beta_1, \beta_2, \dots)$ since this coordinate representation depends only on w.

Let $H \subset L$ be the set of $w \in L$ with $w = (m_1/1 \cdot 2, m_2/2 \cdot 2^2, m_3/3 \cdot 2^3, \dots)$, each m_n an integer, $-2^{n-1} < m_n \leqslant 2^{n-1}$, and $m_{n+1} \equiv m_n \pmod{2^n}$. H is a closed subgroup of L and in fact is a subgroup of a projective limit group, the 2-adic integers, but has a different topology inherited from L. If w and the m_n are as above, define a function σ on H by

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$\sigma(w) = (m_1, m_2, \dots)$. Define for $w \in H$, $\|w\| = \sum_{n=1}^{\infty} |m_n|/n2^n$.

Define $H_n = \{w \in H | w = (\beta_1, \beta_2, \dots)$ with $\beta_i = 0$ if $i \leqslant n\}$. Then the H_n are open subgroups of H and the intersection of all the H_n is the identity.

If $w \in H$ with $\sigma(w) = (m_1, m_2, \dots)$, then for each n, $m_{n+1} = m_n + a_n 2^n$ with $a_n = 0$, 1 or -1. Define $a_0 = m_1$. Then $m_{n+1} = \sum_{r=0}^{n} a_r 2^r$. Note that if r is the least integer with $a_r \neq 0$, then $a_r = 1$ since m_{r+1} cannot be -2^r by definition of the coefficients m_n.

This induces a function θ on H defined by $\theta(w) = (a_0, a_1, \dots)$ for $w \in H$. If $\theta(w) = (a_0, a_1, \dots)$ define $\|w\|'$ by $\|w\|' = \sum_{r=0}^{\infty} |a_r|/(r+1)$. Then if $\sigma(w) = (m_1, m_2, \dots)$,

$$\begin{aligned}
\|w\| &= \sum_{n=1}^{\infty} \frac{|m_n|}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} \left| \sum_{r=0}^{n-1} a_r 2^r \right| \leqslant \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{n-1} |a_r| 2^r \\
&= \sum_{r=0}^{\infty} |a_r| 2^r \sum_{n=r+1}^{\infty} \frac{1}{n2^n} = \sum_{r=0}^{\infty} |a_r| \sum_{n=r+1}^{\infty} \frac{1}{n2^{n-r}} \\
&= \sum_{r=0}^{\infty} |a_r| \sum_{k=1}^{\infty} \frac{1}{k+r} \frac{1}{2^k} \leqslant \sum_{r=0}^{\infty} \frac{|a_r|}{r+1} \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{r=0}^{\infty} \frac{|a_r|}{r+1} = \|w\|'.
\end{aligned}$$

Thus,

$$\|w\| \leqslant \|w\|'. \tag{2.1}$$

On the other hand,

$$\begin{aligned}
\|w\|' &= \sum_{r=0}^{\infty} \frac{|a_r|}{r+1} \leqslant 2 \sum_{r=0}^{\infty} |a_r| \sum_{k=1}^{\infty} \frac{1}{r+k} \frac{1}{2^k} \\
&= 2 \sum_{r=0}^{\infty} |a_r| \sum_{n=r+1}^{\infty} \frac{1}{n} \frac{1}{2^{n-r}} = 2 \sum_{r=0}^{\infty} |a_r| 2^r \sum_{n=r+1}^{\infty} \frac{1}{n2^n} \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{n-1} |a_r| 2^r.
\end{aligned}$$

Thus,

$$\|w\|' \leqslant 2 \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{n-1} |a_r| 2^r. \tag{2.2}$$

Suppose $m_{p+1} \neq 0$ and let q be the greatest q with $q \leqslant p$ and $a_q \neq 0$. Then

$$|m_{p+1}| = |a_q 2^q + m_q| \geqslant |a_q 2^q| - |m_q| \geqslant 2^q - 2^{q-1} = 2^{q-1}.$$

This implies

$$\sum_{r=0}^{p} |a_r| 2^r = \sum_{r=0}^{q} |a_r| 2^r \leqslant \sum_{r=0}^{q} 2^r \leqslant 2^{q+1} \leqslant 4|m_{p+1}|. \tag{2.3}$$

Combining (2.2) with (2.3) gives

$$\|w\|' \leqslant 2 \sum_{n=1}^{\infty} \frac{4|m_n|}{n2^n} = 8\|w\|.$$

Combining this with (2.1) we get $\frac{1}{8}\|w\|' \leqslant \|w\| \leqslant \|w\|'$. Whence $\| \ \|$ and $\| \ \|'$ yield the same topology on H.

Let $w \in H$, $w \neq 0$, and $\theta(w) = (a_0, a_1, \dots)$. If k is an integer with $0 \leqslant k < n$, then

$$2^k m_n = 2^k \sum_{r=0}^{n-1} a_r 2^r = \sum_{r=0}^{n-1} a_r 2^{k+r} \equiv \sum_{r=0}^{n-k-1} a_r 2^{k+r} \pmod{2^n}.$$

Choose p to be the least integer with $a_p \neq 0$. Define s so that $sa_p = |a_p|$. Thus if $y = 2^k w$, then $\theta(y) = (b_0, b_1, \dots)$ with $b_j = 0$ if $j < k$ and $b_j = sa_{j-k}$ if $j \geqslant k$. Whence

$$\|y\|' = \sum_{j=0}^{\infty} \frac{|b_j|}{j+1} = \sum_{j=k}^{\infty} \frac{|b_j|}{j+1} = \sum_{j=k}^{\infty} \frac{|sa_{j-k}|}{j+1} = \sum_{r=0}^{\infty} \frac{|a_r|}{r+k+1}.$$

Given $\epsilon > 0$, choose N so that $\sum_{r=N}^{\infty} |a_r|/(r+1) < \epsilon/2$. Choose t to be a positive integer with $1/t < \epsilon/2$. If $k \geqslant Nt$, then

$$\|y\|' = \sum_{r=0}^{N-1} \frac{|a_r|}{r+k+1} + \sum_{r=N}^{\infty} \frac{|a_r|}{r+k+1} < \frac{N}{Nt} + \frac{\epsilon}{2} < \epsilon.$$

This proves if $w \in H$ with $w \neq 0$, then the subgroup generated by w is not discrete.

Again suppose $w \in H$, $w \neq 0$, and let H_1 be the closed subgroup generated by w. Suppose $\theta(w) = (a_0, a_1, \dots)$ and p is the least integer with $a_p \neq 0$. Given an integer $n > p$, choose s with $s = -1$ if p is odd and $s = 1$ if p is even, and choose an integer q_n so that

$$q_n \sum_{r=p}^{n-1} a_r 2^r \equiv s \sum_{k=p}^{n-1} (-1)^k 2^k \pmod{2^n}.$$

If $y_n = q_n w$, then $\theta(y_n) = (b_0, b_1, \dots)$ with $b_k = s(-1)^k$ if $p \leqslant k < n$ and $b_k = 0$ if $k < p$. Thus $\|y_n\|' \geqslant \sum_{k=p}^{n-1} 1/(k+1)$.

If H_1 is compact, then $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging to some $y \in H_1$. But $\lim_{n\to\infty} \|y_n\|' = \infty$ which contradicts $\|y\|' < \infty$.

Whence the only subgroup F of H such that F is discrete or compact is the trivial subgroup. By the structure theorem for locally compact abelian groups, it follows that the only locally compact subgroup of H is the trivial subgroup. H does have some nontrivial, proper subgroups, e.g., the H_n.

This example leaves unanswered the question of whether, if H is an infinite, complete, metrizable topological abelian group, H has a nontrivial, proper closed subgroup.

3. **Some groups which have nontrivial locally compact subgroups.** If C_0 is the Banach space of real sequences which converge to zero and K the group of integer sequences with only a finite number of nonzero terms, define G to be the quotient group C_0/K [1]. If H is a nontrivial, closed subgroup of G, then we will see that H has a nontrivial locally compact subgroup. This will be a consequence of a more general result for Banach spaces which have a basis.

Let $\{u_n\}_{n=1}^{\infty}$ be a basis for a Banach space $(E, \| \ \|'')$ with $\|u_n\|'' = 1$ for all

n. There is a norm $\| \ \|'$ on E equivalent to $\| \ \|''$ such that

$$\left\| \sum_{n=1}^{\infty} a_n u_n \right\|' = \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|'' \tag{3.1}$$

if $\sum_{n=1}^{\infty} a_n u_n \in E$ [2, p. 180, Theorem 19.1(c)]. By (3.1), $\|u_n\|' = 1$. In general, if $\sum_{n=1}^{\infty} a_n u_n \in E$, then

$$\left\| \sum_{n=1}^{\infty} a_n u_n \right\|' = \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|'' = \sup_{m,p} \left(\sup_{m \leqslant k \leqslant l \leqslant p} \left\| \sum_{n=k}^{l} a_n u_n \right\|'' \right)$$
$$= \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|'.$$

Thus,

$$\left\| \sum_{n=1}^{\infty} a_n u_n \right\|' = \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|'. \tag{3.2}$$

Let $K = \{x \in E | x = \sum_{n=1}^{\infty} s_n u_n \text{ with } s_n \in Z\}$. K is a discrete subgroup since (3.1) implies if $x \neq 0$ and $x = \sum_{n=1}^{\infty} s_n u_n$ with $s_n \in Z$, then $\|x\|' \geqslant \max_n \|s_n u_n\|'' \geqslant 1$. Also, if $x = \sum_{n=1}^{\infty} a_n u_n$, then $\lim_{n\to\infty} a_n = 0$ [2, p. 20, Lemma 3.1]. Thus if $x = \sum_{n=1}^{\infty} s_n u_n$ is in K, then $s_n \in Z$ and for some N, $x = \sum_{n=1}^{N} s_n u_n$.

Let G be the quotient group $G = E/K$ and π: $E \to G$ be the quotient map. Define $\|y\|$ for $y \in G$ to be the minimum of $\|x\|'$ over all $x \in E$ with $\pi(x) = y$. If $x \in E$, we can write $x = \sum_{n=1}^{\infty} x_n u_n + w$ with $w \in K$, $x_n \in R$ and $-\frac{1}{2} < x_n \leqslant \frac{1}{2}$. If $y = \pi(x)$, we will write $y = (y_1, y_2, \dots)$ with $y_n = x_n$ for the x_n just described. In general, if $y, z \in G$, then $\|y + z\| \leqslant \|y\| + \|z\|$ and $\|y\| = \|-y\|$.

LEMMA . *Suppose H is a closed subgroup of G. Also suppose that, given a positive integer $n > 1$ and positive real numbers ϵ and M, there is $b = (b_1, b_2, \dots)$ in H with $0 < \|b\| \leqslant \epsilon$ and with $\|(b_1, b_2, \dots, b_n, 0, 0, \dots)\| \leqslant M\|b\|$. Then H contains an infinite, discrete subgroup.*

PROOF. Choose $b^1 = (b_1^1, b_2^1, \dots)$ in H such that $0 < \|b^1\| \leqslant \frac{1}{2}$. Then for some positive integer r, $\frac{1}{4} < |r| \, \|b^1\| \leqslant \frac{1}{2}$. Define $c^1 = rb^1$. Write $c^1 = (c_1^1, c_2^1, \dots)$. Define $k(1) = 0$ and choose $k(2) > 0$ so that

$$\|(0, 0, \dots, 0, c_{k(2)+1}^1, c_{k(2)+2}^1, \dots)\| < 1/2 \cdot 16 \cdot 2^2.$$

Inductively suppose $\{c^m\}_{m=1}^{n} \subset H$ and $\{k(m)\}_{m=1}^{n+1}$ are given such that $m_1 < m_2$ implies $k(m_1) < k(m_2)$, $1/2 \cdot 2^m < \|c^m\| \leqslant 1/2^m$, $m < m_1 \leqslant n + 1$ implies

$$\|(0, 0, \dots, 0, c_{k(m_1)+1}^m, c_{k(m_1)+2}^m, \dots)\| < 1/m_1 \cdot 16 \cdot 2^{m_1},$$

and that $m \leqslant n$ implies

$$\|(c_1^m, c_2^m, \dots, c_{k(m)}^m, 0, 0, \dots)\| < 1/2^{2m} \cdot 16.$$

By the hypothesis of the lemma, we can find $b^{n+1} = (b_1^{n+1}, b_2^{n+1}, \dots)$ in H with $0 < \|b^{n+1}\| \leqslant 1/2^{n+1}$ and

$$\|(b_1^{n+1}, b_2^{n+1}, \dots, b_{k(n+1)}^{n+1}, 0, 0, \dots)\| \leqslant \|b^{n+1}\|/2^{n+1} \cdot 16.$$

Choose a positive integer r so that $1/2 \cdot 2^{n+1} < |r|\,\|b^{n+1}\| \leqslant 1/2^{n+1}$. Define $c^{n+1} = rb^{n+1}$. Choose $k(n+2)$ so that $k(n+1) < k(n+2)$ and $m < n+2$ implies

$$\|(0, 0, \dots, 0, c_{k(n+2)+1}^m, c_{k(n+2)+2}^m, \dots)\| < 1/(n+2) \cdot 16 \cdot 2^{n+2}.$$

Note that

$$\|(0, 0, \dots, 0, c_{k(n)+1}^n, c_{k(n)+2}^n, \dots, c_{k(n+1)}^n, 0, 0, \dots)\| \geqslant \tfrac{1}{2}\|c^n\|$$

for all n.

Define $a \in H$ by $a = \sum_{p=1}^{\infty} c^p$. $\sum_{p=1}^{\infty} c^p$ makes sense since $\sum_{p=1}^{\infty} \|c^p\| \leqslant \sum_{p=1}^{\infty} 1/2^p = 1$ and since H is complete. Write $a = (a_1, a_2, \dots)$.

Given an integer $q \neq 0$, choose $n \geqslant 0$ so that $2^n \leqslant q < 2^{n+1}$. Then by (3.2), the definition of K and $\|\ \|$, we get

$$\begin{aligned}
\|qa\| &\geqslant \|q(0, \dots, 0, a_{k(n+2)+1}, \dots, a_{k(n+3)}, 0, 0, \dots)\| \\
&\geqslant |q|\Bigg(\|(0, \dots, 0, c_{k(n+2)+1}^{n+2}, \dots, c_{k(n+3)}^{n+2}, 0, 0, \dots)\| \\
&\qquad - \sum_{m=1}^{n+1} \|(0, \dots, 0, c_{k(n+2)+1}^m, \dots, c_{k(n+3)}^m, 0, 0, \dots)\| \\
&\qquad - \sum_{m=n+3}^{\infty} \|(0, \dots, 0, c_{k(n+2)+1}^m, \dots, c_{k(n+3)}^m, 0, 0, \dots)\|\Bigg) \\
&\geqslant 2^n\left(\frac{1}{2} \cdot \frac{1}{2 \cdot 2^{n+2}} - \sum_{m=1}^{n+1} \frac{1}{n+2} \cdot \frac{1}{16 \cdot 2^{n+2}} - \sum_{m=n+3}^{\infty} \frac{1}{2^{2m} \cdot 2}\right) \\
&\geqslant 1/16 - 1/64 - 1/64 = 1/16 - 1/32 = 1/32.
\end{aligned}$$

Thus a generates a discrete subgroup of H.

THEOREM. *Let $(E, \|\ \|')$ be a Banach space with a basis $\{u_n\}_{n=1}^{\infty}$ such that $\|u_n\|' = 1$ for all n. Also let K be the subgroup generated by $\{u_n\}_{n=1}^{\infty}$ and $G = E/K$. If H is a nontrivial, closed subgroup of G, then either H has a nontrivial, compact subgroup or H has an infinite cyclic, discrete subgroup.*

PROOF. Suppose H is a nontrivial, closed subgroup of G whose only discrete subgroup is the trivial group. Then by the lemma, there is an integer $n > 0$ and real numbers $\epsilon > 0$ and $M > 0$ such that there is no $b = (b_1, b_2, \dots)$ in H with $0 < \|b\| \leqslant \epsilon$ and $\|(b_1, \dots, b_n, 0, \dots)\| \leqslant M\|b\|$.

Define $L = \{x \in G \mid x = (x_1, x_2, \dots)$ and $x_i = 0$ if $i > n\}$ and define θ: $G \to L$ to be the projection with $\theta(y) = (z_1, z_2, \dots)$ for $y = (y_1, y_2, \dots)$, $z_i = y_i$ if $i \leqslant n$ and $z_i = 0$ if $i > n$. Also define $U = \{x \in H \mid \|x\| \leqslant \epsilon/2\}$.

Suppose $\{x^i\}_{i=1}^{\infty}$ is in U and $\{\theta(x_i)\}_{i=1}^{\infty}$ is a nonrepeating Cauchy sequence

in L. Then if $\epsilon_1 > 0$ is given, there is $N(\epsilon_1 M)$ such that $i, j > N(\epsilon_1 M)$ implies $\|\theta(x^i) - \theta(x^j)\| < \epsilon_1 M$. But we know since $\|x^i - x^j\| \leqslant \epsilon$ that $\|\theta(x^i - x^j)\| \geqslant M\|x^i - x^j\|$. Thus, $\epsilon_1 M > \|\theta(x^i) - \theta(x^j)\| \geqslant M\|x^i - x^j\|$ and $\|x^i - x^j\| < \epsilon_1$ so that $\{x^i\}_{i=1}^{\infty}$ is a Cauchy sequence in U.

If $\{y^j\}_{j=1}^{\infty}$ is any sequence in U, then $\{\theta(y^j)\}_{j=1}^{\infty}$ has a convergent subsequence $\{\theta(y^{j(i)})\}_{i=1}^{\infty}$ since L is compact. The preceding result shows $\{y^{j(i)}\}_{i=1}^{\infty}$ is a Cauchy sequence in U. U is a complete metric space under the metric induced from $\| \ \|$ since $G = E/K$ is complete, H is closed in G, and U is closed in H. Whence $\{y^{j(i)}\}_{i=1}^{\infty}$ converges to some $y \in U$. This proves that U is compact. Whence H is a locally compact topological group. Then the structure theorem for locally compact abelian groups implies H has an infinite cyclic, discrete subgroup or a nontrivial compact subgroup.

It is not known whether this theorem holds true if the hypotheses of the theorem are weakened so that E is any separable Banach space and K is a discrete subgroup of E.

Let E be a separable Banach space and K a discrete subgroup. K is said to have the maximal property if K_1 is a discrete subgroup and $K_1 \supset K$ imply that K_1/K is a torsion group. If K is a discrete subgroup of a Banach space E such that K does not have the maximal property, then E/K has an infinite, discrete subgroup. If E is finite dimensional of dimension n, then any set of n elements which are linearly independent over the reals generate a subgroup K with the maximal property. No example is known of such a pair E, K if E is infinite dimensional.

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