LOCALLY COMPACT SUBGROUPS OF
METRIZABLE TOPOLOGICAL ABELIAN
GROUPS

R. C. HOOKER

ABSTRACT. This paper examines the question of existence of locally compact subgroups of topological abelian groups. A counterexample is given and it is shown in a certain setting that groups have a nontrivial locally compact subgroup.

1. Introduction. If $H$ is an infinite, locally compact topological abelian group, then $H$ has a nontrivial, proper locally compact subgroup $H_1$. More generally, suppose $H$ is an infinite, complete, metrizable, topological abelian group. We will see that $H$ may not have any nontrivial, locally compact subgroups.

When such a topological group $H$ is a subgroup of a quotient group of the Banach space $C_0$ modulo the integer valued sequences, $H$ will have a nontrivial locally compact subgroup. This will follow from a more general result involving Banach space with bases. The results in this paper are established by constructions utilizing coordinates.

2. A group with no nontrivial locally compact subgroups. The following construction will yield an example of an infinite, complete, metrizable topological group $H$ whose only locally compact subgroup is the trivial one.

Let $\{u_n\}_{n=1}^{\infty}$ be the standard basis for $l_1$, $u_n = (1/n)u_n$, and $K$ be the closed subgroup of $l_1$ generated by $\{u_n\}_{n=1}^{\infty}$. Thus if $x \in K$, we can write $x$ in standard $l_1$ coordinates as $x = (p_1/1, p_2/2, p_3/3, \ldots)$ with $\sum_{n=1}^{\infty} |p_n|/n < \infty$ and the $p_n$ integers. If $y \in l_1$ with $y = (a_1, a_2, \ldots)$, we can write $y = x + z$ with $x \in K$, $z \in l_1$, $z = (\beta_1, \beta_2, \ldots)$ and $-1/2n < \beta_n \leq 1/2n$ for each $n$.

Define $L$ to be the quotient group $L = l_1/K$. If $w \in L$, $w$ is the image under the quotient map of some $y \in l_1$ with $y$ as above, $y = x + z$, etc. We will represent $w$ as $w = (\beta_1, \beta_2, \ldots)$ since this coordinate representation depends only on $w$.

Let $H \subset L$ be the set of $w \in L$ with $w = (m_1/1 \cdot 2, m_2/2 \cdot 2^2, m_3/3 \cdot 2^3, \ldots)$, each $m_n$ an integer, $-2^{n-1} < m_n \leq 2^{n-1}$, and $m_{n+1} \equiv m_n \pmod{2^n}$. $H$ is a closed subgroup of $L$ and in fact is a subgroup of a projective limit group, the 2-adic integers, but has a different topology inherited from $L$. If $w$ and the $m_n$ are as above, define a function $\sigma$ on $H$ by

Received by the editors July 1, 1975.


Key words and phrases. Topological group, locally compact subgroup, closed subgroups of Banach spaces.

© American Mathematical Society 1976

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
$\sigma(w) = (m_1, m_2, \ldots)$. Define for $w \in H$, $\|w\| = \sum_{n=1}^{\infty} |m_n|/n2^n$.

Define $H_n = \{w \in H | w = (\beta_1, \beta_2, \ldots) \text{ with } \beta_i = 0 \text{ if } i < n \}$. Then the $H_n$ are open subgroups of $H$ and the intersection of all the $H_n$ is the identity.

If $w \in H$ with $\sigma(w) = (m_1, m_2, \ldots)$, then for each $n$, $m_{n+1} = m_n + a_n 2^n$ with $a_n = 0, 1 \text{ or } -1$. Define $a_0 = m_1$. Then $m_{n+1} = \sum_{r=0}^{n-1} a_r 2^r$.

Note that if $r$ is the least integer with $a_r \neq 0$, then $a_r = 1$ since $m_{r+1}$ cannot be $-2^r$ by definition of the coefficients $m_n$.

This induces a function $\theta$ on $H$ defined by $\theta(w) = (a_0, a_1, \ldots)$ for $w \in H$. If $\theta(w) = (a_0, a_1, \ldots)$ define $\|w\|^\prime$ by $\|w\|^\prime = \sum_{r=0}^{\infty} |a_r|/(r + 1)$. Then if $\sigma(w) = (m_1, m_2, \ldots)$,

$$\|w\| = \sum_{n=1}^{\infty} \frac{|m_n|}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n2^n} \left| \sum_{r=0}^{n-1} a_r 2^r \right| \leq \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{n-1} |a_r| 2^r = \sum_{r=0}^{\infty} |a_r| 2^r \sum_{n=r+1}^{\infty} \frac{1}{n2^n} = \sum_{r=0}^{\infty} \frac{|a_r|}{r + 1} = \|w\|^\prime.$$  

Thus,

(2.1) $\|w\| \leq \|w\|^\prime$.

On the other hand,

$$\|w\|^\prime = \sum_{r=0}^{\infty} \frac{|a_r|}{r + 1} \leq 2 \sum_{r=0}^{\infty} |a_r| \sum_{k=1}^{\infty} \frac{1}{r + k} \frac{1}{2^k} = 2 \sum_{r=0}^{\infty} |a_r| \sum_{n=r+1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{\infty} \frac{1}{n2^n} = 2 \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{n-1} |a_r| 2^r.$$  

Thus,

(2.2) $\|w\|^\prime \leq 2 \sum_{n=1}^{\infty} \frac{1}{n2^n} \sum_{r=0}^{n-1} |a_r| 2^r$.

Suppose $m_{p+1} \neq 0$ and let $q$ be the greatest $q$ with $q \leq p$ and $a_q \neq 0$. Then

$$|m_{p+1}| = |a_q 2^q + m_q| \geq |a_q 2^q| - |m_q| \geq 2^q - 2^{q-1} = 2^{q-1}.$$  

This implies

(2.3) $\sum_{r=0}^{p} |a_r| 2^r = \sum_{r=0}^{q} |a_r| 2^r \leq \sum_{r=0}^{q} 2^r \leq 2^{q+1} \leq 4|m_{p+1}|$.

Combining (2.2) with (2.3) gives

$$\|w\|^\prime \leq 2 \sum_{n=1}^{\infty} \frac{4|m_n|}{n2^n} = 8\|w\|.$$
Combining this with (2.1) we get $\frac{1}{2} ||w||' \leq ||w|| \leq ||w||'$. Whence $||\cdot||$ and $||\cdot||'$ yield the same topology on $H$.

Let $w \in H$, $w \neq 0$, and $\theta(w) = (a_0, a_1, \ldots)$. If $k$ is an integer with $0 \leq k < n$, then

$$2^k m_n = 2^k \sum_{r=0}^{n-1} a_r 2^r = \sum_{r=0}^{n-1} a_r 2^{k+r} \equiv \sum_{r=0}^{n-k-1} a_r 2^{k+r} \pmod{2^n}.$$  

Choose $p$ to be the least integer with $a_p \neq 0$. Define $s$ so that $s a_p = |a_p|$. Thus if $y = 2^k w$, then $\theta(y) = (b_0, b_1, \ldots)$ with $b_j = 0$ if $j < k$ and $b_j = s a_{j-k}$ if $j \geq k$. Whence

$$\|y\|' = \sum_{j=0}^{\infty} |b_j| = \sum_{j=k}^{\infty} |b_j| = \sum_{j=k}^{\infty} |s a_{j-k}| = \sum_{r=0}^{\infty} |a_r| = \sum_{r=0}^{\infty} \frac{|a_r|}{r + k + 1}.$$  

Given $\epsilon > 0$, choose $N$ so that $\sum_{r=0}^{\infty} \frac{|a_r|}{r + 1} < \epsilon/2$. Choose $t$ to be a positive integer with $1/t < \epsilon/2$. If $k \geq Nt$, then

$$\|y\|' = \sum_{r=0}^{N-1} \frac{|a_r|}{r + k + 1} + \sum_{r=N}^{\infty} \frac{|a_r|}{r + k + 1} < \frac{N}{Nt} + \frac{\epsilon}{2}.$$  

This proves if $w \in H$ with $w \neq 0$, then the subgroup generated by $w$ is not discrete.

Again suppose $w \in H$, $w \neq 0$, and let $H_1$ be the closed subgroup generated by $w$. Suppose $\theta(w) = (a_0, a_1, \ldots)$ and $p$ is the least integer with $a_p \neq 0$. Given an integer $n > p$, choose $s$ with $s = -1$ if $p$ is odd and $s = 1$ if $p$ is even, and choose an integer $q_n$ so that

$$q_n \sum_{r=p}^{n-1} a_r 2^r = s \sum_{k=p}^{n-1} (-1)^k 2^k \pmod{2^n}.$$  

If $y_n = q_n w$, then $\theta(y_n) = (b_0, b_1, \ldots)$ with $b_k = s(-1)^k$ if $p \leq k < n$ and $b_k = 0$ if $k < p$. Thus $\|y_n\|' \geq \sum_{k=p}^{n-1} 1/(k + 1)$.

If $H_1$ is compact, then $\{y_n\}_{n=1}^{\infty}$ has a subsequence converging to some $y \in H_1$. But $\lim_{n \to \infty} \|y_n\|' = \infty$ which contradicts $\|y\|' < \infty$.

Whence the only subgroup $F$ of $H$ such that $F$ is discrete or compact is the trivial subgroup. By the structure theorem for locally compact abelian groups, it follows that the only locally compact subgroup of $H$ is the trivial subgroup. $H$ does have some nontrivial, proper subgroups, e.g., the $H_n$.

This example leaves unanswered the question of whether, if $H$ is an infinite, complete, metrizable topological abelian group, $H$ has a nontrivial, proper closed subgroup.

3. Some groups which have nontrivial locally compact subgroups. If $C_0$ is the Banach space of real sequences which converge to zero and $K$ the group of integer sequences with only a finite number of nonzero terms, define $G$ to be the quotient group $C_0/K [1]$. If $H$ is a nontrivial, closed subgroup of $G$, then we will see that $H$ has a nontrivial locally compact subgroup. This will be a consequence of a more general result for Banach spaces which have a basis.

Let $\{u_n\}_{n=1}^{\infty}$ be a basis for a Banach space $(E, \|\cdot\|')$ with $\|u_n\|' = 1$ for all
There is a norm \( \| \|' \) on \( E \) equivalent to \( \| \|'' \) such that

\[
(3.1) \quad \left\| \sum_{n=1}^{\infty} a_n u_n \right\|'' = \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|''
\]

if \( \sum_{n=1}^{\infty} a_n u_n \in E \) [2, p. 180, Theorem 19.1(c)]. By (3.1), \( \|u_n\|' = 1 \). In general, if \( \sum_{n=1}^{\infty} a_n u_n \in E \), then

\[
\left\| \sum_{n=1}^{\infty} a_n u_n \right\|'' = \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|'' = \sup_{m,p} \left( \sup_{m \leq k \leq l \leq p} \left\| \sum_{n=k}^{l} a_n u_n \right\|'' \right)
\]

Thus,

\[
(3.2) \quad \left\| \sum_{n=1}^{\infty} a_n u_n \right\|' = \sup_{m,p} \left\| \sum_{n=m}^{p} a_n u_n \right\|'.
\]

Let \( K = \{ x \in E | x = \sum_{n=1}^{\infty} s_n u_n \text{ with } s_n \in \mathbb{Z} \} \). \( K \) is a discrete subgroup since (3.1) implies if \( x \neq 0 \) and \( x = \sum_{n=1}^{\infty} s_n u_n \) with \( s_n \in \mathbb{Z} \), then \( \|x\|' > \max_n \|s_n u_n\|'' > 1 \). Also, if \( x = \sum_{n=1}^{\infty} a_n u_n \), then \( \lim_{n \to \infty} a_n = 0 \) [2, p. 20, Lemma 3.1]. Thus if \( x = \sum_{n=1}^{\infty} s_n u_n \) is in \( K \), then \( s_n \in \mathbb{Z} \) and for some \( N, x = \sum_{n=1}^{N} s_n u_n \).

Let \( G \) be the quotient group \( G = E/K \) and \( \pi: E \to G \) be the quotient map. Define \( \|y\| \) for \( y \in G \) to be the minimum of \( \|x\|' \) over all \( x \in E \) with \( \pi(x) = y \). If \( x \in E \), we can write \( x = \sum_{n=1}^{\infty} x_n u_n + w \) with \( w \in K, x_n \in \mathbb{R} \) and \( -\frac{1}{2} < x_n \leq \frac{1}{2} \). If \( y = \pi(x) \), we will write \( y = (y_1, y_2, \ldots) \) with \( y_n = x_n \) for the \( x_n \) just described. In general, if \( y, z \in G \), then \( \|y + z\| \leq \|y\| + \|z\| \) and \( \|y\| = \|-y\| \).

Lemma : Suppose \( H \) is a closed subgroup of \( G \). Also suppose that, given a positive integer \( n \geq 1 \) and positive real numbers \( \epsilon \) and \( M \), there is \( b = (b_1, b_2, \ldots) \in H \) with \( 0 < \|b\| < \epsilon \) and with \( \|b_1, b_2, \ldots, b_n, 0, 0, \ldots\| \leq M \|b\| \).

Then \( H \) contains an infinite, discrete subgroup.

Proof. Choose \( b^1 = (b_1^1, b_2^1, \ldots) \in H \) such that \( 0 < \|b^1\| \leq \frac{1}{2} \). Then for some positive integer \( r, \frac{1}{4} < |r| \|b^1\| \leq \frac{1}{2} \). Define \( c^1 = rb^1 \). Write \( c^1 = (c_1^1, c_2^1, \ldots) \). Define \( k(1) = 0 \) and choose \( k(2) > 0 \) so that

\[
\left\| (0,0,\ldots,0,c_{k(2)+1}^1,c_{k(2)+2}^1,\ldots) \right\| < \frac{1}{2} \cdot 16 \cdot 2^2.
\]

Inductively suppose \( \{c^m\}_{m=1}^{n} \subset H \) and \( \{k(m)\}_{m=1}^{n+1} \) are given such that \( m_1 < m_2 \) implies \( k(m_1) < k(m_2) \), \( 1/2 \cdot 2^m \leq \|c^m\| \leq 2^m, m = m_1 < n + 1 \) implies

\[
\left\| (0,0,\ldots,0,c_{k(m_1)+1}^m,c_{k(m_1)+2}^m,\ldots) \right\| < m_1 \cdot 16 \cdot 2^{m_1},
\]

and that \( m \leq n \) implies

\[
\left\| (c_1^m,c_2^m,\ldots,c_{k(m)}^m,0,0,\ldots) \right\| < 1/2^{2m} \cdot 16.
\]
By the hypothesis of the lemma, we can find $b^{n+1} = (b_1^{n+1}, b_2^{n+1}, \ldots)$ in $H$ with $0 < \| b^{n+1} \| \leq 1/2^{n+1}$ and

$$
\|(b_1^{n+1}, b_2^{n+1}, \ldots, b_{k(n+1)}^{n+1}, 0, 0, \ldots)\| \leq \| b^{n+1} \|/2^{n+1} \cdot 16.
$$

Choose a positive integer $r$ so that $1/2 \cdot 2^{n+1} \leq |r| \| b^{n+1} \| \leq 1/2^{n+1}$. Define $c^{n+1} = r b^{n+1}$. Choose $k(n+2)$ so that $k(n+1) < k(n+2)$ and $m < n + 2$ implies

$$
\|(0, 0, \ldots, 0, c_k^{m(n+2)+1}, c_{k(n+2)+2}, \ldots)\| < 1/(n + 2) \cdot 16 \cdot 2^{n+2}.
$$

Note that

$$
\|(0, 0, \ldots, 0, c_k^{n(n+2)+1}, c_{k(n+2)+2}, \ldots, c_k^{n(n+1)+1}, 0, 0, \ldots)\| \geq \frac{1}{2} \| c^n \|
$$

for all $n$.

Define $a \in H$ by $a = \sum_{p=1}^{\infty} c^p \cdot \sum_{p=1}^{\infty} c^p$ makes sense since $\sum_{p=1}^{\infty} \| c^p \| < \sum_{p=1}^{\infty} 1/2^p = 1$ and since $H$ is complete. Write $a = (a_1, a_2, \ldots)$.

Given an integer $q \neq 0$, choose $n > 0$ so that $2^n < q < 2^{n+1}$. Then by (3.2), the definition of $A$ and $\| \|$ we get

$$
\|qa\| \geq \|q(0, \ldots, 0, a_{k(n+2)+1}, \ldots, a_{k(n+3)}, 0, 0, \ldots)\|
$$

$$
\geq |q| \left(\|(0, 0, \ldots, 0, c_k^{n+1}, \ldots, c_k^{n+2}, c_k^{n+1}, 0, 0, \ldots)\| - \sum_{m=1}^{n+1} \|(0, 0, \ldots, 0, c_k^{m(n+2)+1}, \ldots, c_k^{m(n+3)}, 0, 0, \ldots)\| - \sum_{m=n+3}^{\infty} \|(0, 0, \ldots, 0, c_k^{m(n+2)+1}, \ldots, c_k^{m(n+3)}, 0, 0, \ldots)\|\right)
$$

$$
\geq 2^n \left(\frac{1}{2} - \frac{1}{2^{n+2}} - \frac{1}{n + 2} \cdot \frac{1}{16 \cdot 2^{n+2}} - \frac{1}{n+3} \cdot \frac{1}{2^{2m} \cdot 2}\right)
$$

$$
\geq 1/16 - 1/64 - 1/64 = 1/16 - 1/32 = 1/32.
$$

Thus $a$ generates a discrete subgroup of $H$.

**Theorem.** Let $(E, \| \|')$ be a Banach space with a basis $\{u_n\}_{n=1}^{\infty}$ such that $\|u_n\|' = 1$ for all $n$. Also let $K$ be the subgroup generated by $\{u_n\}_{n=1}^{\infty}$ and $G = E/K$. If $H$ is a nontrivial, closed subgroup of $G$, then either $H$ has a nontrivial, compact subgroup or $H$ has an infinite cyclic, discrete subgroup.

**Proof.** Suppose $H$ is a nontrivial, closed subgroup of $G$ whose only discrete subgroup is the trivial group. Then by the lemma, there is an integer $n > 0$ and real numbers $\epsilon > 0$ and $M > 0$ such that there is no $b = (b_1, b_2, \ldots)$ in $H$ with $0 < \| b \| \leq \epsilon$ and $\| (b_1, \ldots, b_n, 0, \ldots) \| \leq M \| b \|$.

Define $L = \{x \in G | x = (x_1, x_2, \ldots) \text{ and } x_i = 0 \text{ if } i > n \}$ and define $\theta: G \to L$ to be the projection with $\theta(y) = (z_1, z_2, \ldots)$ for $y = (y_1, y_2, \ldots)$, $z_i = y_i$ if $i \leq n$ and $z_i = 0$ if $i > n$. Also define $U = \{x \in H | \| x \|' \leq \epsilon/2 \}$.

Suppose $\{x_i\}_{i=1}^{\infty}$ is in $U$ and $\{\theta(x_i)\}_{i=1}^{\infty}$ is a nonrepeating Cauchy sequence
in $L$. Then if $\epsilon_1 > 0$ is given, there is $N(\epsilon_1 M)$ such that $i, j > N(\epsilon_1 M)$ implies $\|\theta(x^i) - \theta(x^j)\| < \epsilon_1 M$. But we know since $\|x^i - x^j\| \leq \epsilon$ that $\|\theta(x^i) - x^i - \theta(x^j) - x^j\| \geq M\|x^i - x^j\|$. Thus, $\epsilon_1 M > \|\theta(x^i) - \theta(x^j)\| \geq M\|x^i - x^j\|$ and $\|x^i - x^j\| < \epsilon_1$ so that $\{x^i\}_{i=1}^\infty$ is a Cauchy sequence in $U$.

If $\{y^j\}_{j=1}^\infty$ is any sequence in $U$, then $\{(\theta(y^j))_{i=1}^\infty\}$ has a convergent subsequence $\{(\theta(y^{j(i)}))_{i=1}^\infty\}$ since $L$ is compact. The preceding result shows $\{y^{j(i)}\}_{i=1}^\infty$ is a Cauchy sequence in $U$. $U$ is a complete metric space under the metric induced from $\|\|$ since $G = E/K$ is complete, $H$ is closed in $G$, and $U$ is closed in $H$. Whence $\{y^{j(i)}\}_{i=1}^\infty$ converges to some $y \in U$. This proves that $U$ is compact. Whence $H$ is a locally compact topological group. Then the structure theorem for locally compact abelian groups implies $H$ has an infinite cyclic, discrete subgroup or a nontrivial compact subgroup.

It is not known whether this theorem holds true if the hypotheses of the theorem are weakened so that $E$ is any separable Banach space and $K$ is a discrete subgroup of $E$.

Let $E$ be a separable Banach space and $K$ a discrete subgroup. $K$ is said to have the maximal property if $K_1$ is a discrete subgroup and $K_1 \subseteq K$ imply that $K_1/K$ is a torsion group. If $K$ is a discrete subgroup of a Banach space $E$ such that $K$ does not have the maximal property, then $E/K$ has an infinite, discrete subgroup. If $E$ is finite dimensional of dimension $n$, then any set of $n$ elements which are linearly independent over the reals generate a subgroup $K$ with the maximal property. No example is known of such a pair $E, K$ if $E$ is infinite dimensional.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEVADA, RENO, NEVADA 89507