PRIMITIVE IDEALS IN GROUP RINGS OF POLYCYCLIC GROUPS

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Abstract. If $F$ is a field which is not algebraic over a finite field and $G$ is a polycyclic group, then all primitive ideals of the group ring $F[G]$ are maximal if and only if $G$ is nilpotent-by-finite.

We recall that a primitive ring is a ring with a faithful irreducible module. An ideal is primitive if the factor ring is a primitive ring.

If $F$ is a field algebraic over a finite field and $G$ a polycyclic group, then Roseblade has recently shown that every irreducible module for the group ring $F[G]$ is finite dimensional [5]. This implies that the primitive factor rings are simple Artin. On the other hand, if $F$ is any other field and $G$ is not abelian-by-finite, then Hall observed that $F[G]$ has infinite dimensional irreducible modules [2]. If $G$ is finitely generated nilpotent, Zalesskiï proved that the primitive factor rings are at least simple for any field $F$ [6]. In this paper we offer a converse to Zalesskiï's theorem by proving the

Theorem. If $F$ is a field which is not algebraic over a finite field and $G$ is a polycyclic group, then all primitive ideals of the group ring $F[G]$ are maximal if and only if $G$ is nilpotent-by-finite.

We warn the prospective reader of Zalesskiï's paper [6] that the word "primitive" has been translated as prime throughout.

Lemma 1. If $G$ is polycyclic and $H$ is a subgroup of finite index with all primitive ideals of $F[H]$ maximal, then all primitive ideals of $F[G]$ are also maximal.


Lemma 2. Let $G$ be polycyclic and $H$ a subgroup of finite index in $G$. If $F[H]$ has a nonmaximal primitive ideal, then $F[G]$ does also.

Proof. By Lemma 1, we may assume $H$ is normal in $G$. Let $P$ be a primitive ideal of $F[H]$ properly contained in a maximal ideal $Q$. Let $1 = g_1, g_2, \ldots, g_n$ be a set of coset representatives for $H$ in $G$. Let $\overline{P} = \bigcap_{i=1}^{n} g_i^{-1}P$ and $\overline{Q} = \bigcap_{i=1}^{n} g_i^{-1}Qg_i^{-1}$. $\overline{P} \subseteq \overline{Q}$ since equality would imply that $P \supseteq g_iQg_i^{-1}$ for some $i$ and hence $P$ would be maximal. Let $V$ be an irreducible $F[H]$ module with annihilator $P$. $\overline{V} = V \otimes_{F[H]} F[G]$ has finite length as an $F[H]$ module and hence as an $F[G]$ module. Let $0 = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = \overline{V}$ be a $F[G]$ composition series for $\overline{V}$. The annihilator of $\overline{V}$ is $\overline{P}G$. $\overline{Q}G$ is a two-sided ideal
and hence can be embedded in a maximal ideal $M$. Let $P_i = \text{Ann} \left( \frac{W_i}{W_{i-1}} \right)$. Since $P_n P_{n-1} \cdots P_1$ annihilates $V$, $P_n P_{n-1} \cdots P_1 \subseteq \overline{FG} \subseteq \overline{QG} \subseteq M$. Hence $M$ contains $P_i$ for some $i$. Now $W_i/W_{i-1}$ contains a copy of $V_{g_j}$ for some $j$ and hence contains a copy of $V_{g_j}$ for each $j$. Hence $P_i \cap F[H] = \text{Ann}_{F[H]} \left( \frac{W_i}{W_{i-1}} \right) = \bigcap_{j=1}^n \text{Ann} \left( V_{g_j} \right) = \overline{P}$. Since $M \cap F[H] \supseteq \overline{QG} \cap F[H] = Q$, we have $P_i \subseteq M$ and $P_i$ is a nonmaximal primitive ideal.

**Proof of Theorem.** If $G$ is nilpotent-by-finite, the result follows from Zalesskiï [6] and Lemma 1. Conversely suppose $G$ is not nilpotent-by-finite. Pick a subgroup $K$ maximal among the subgroups with $N = \cap G(K)$ of finite index and $N/K$ not nilpotent-by-finite. Using Lemma 2, we may assume $K$ is normal. Also since $F[G/K]$ is a homomorphic image of $F[G]$ we assume $K = X$. The finite conjugate subgroup of $G$ is trivial. Otherwise $G$ has a nontrivial normal subgroup $H$ whose centralizer $C$ has finite index. $C/C \cap H \cong CH/H$ is nilpotent-by-finite by the maximality of $K$. Also $C \cap H$ is central in $C$ and hence $C$ and therefore $G$ is nilpotent-by-finite, a contradiction. Let $A$ be a maximal abelian normal subgroup. $G$ contains a subgroup $G_1$ of finite index such that $A$ contains a normal subgroup $B$ of $G_1$ with the property that $G_1$ and all of its subgroups of finite index act rationally irreducibly on $B$ [5, Lemma 2]. By Lemma 2, we assume $G = G_1$. $A$ is torsion free since $G$ has trivial finite conjugate subgroup. The rank of $B$ is at least two for the same reason. We may clearly replace $B$ with $QB \cap A$. $QA$ is a $Q[G/A]$ module and $QB$ is an irreducible submodule. We claim that $QA$ is an essential extension of $QB$. Suppose to the contrary that $T$ is a nontrivial $Q[G/A]$ submodule with $T \cap QB = 0$. $T \cap A$ is a nontrivial normal subgroup of $G$ and $G/(T \cap A)$ is nilpotent-by-finite. This is impossible since $G/T \cap A$ contains an isomorphic copy of $B$ and the rank of $B$ is at least two. Again using Lemma 2, we may assume $G/B$ is nilpotent. Let $U/QB$ be an irreducible $Q[G/A]$ submodule of $QA/QB$. $U \cap A/B$ must intersect the center of $G/B$ nontrivially. Therefore, $U/QB$ must have $Q$ dimension 1. Clearly $\text{Ann} \left( U \right) \subseteq \text{Ann} \left( QB \right)$. If $P_1 = \text{Ann} \left( U/QB \right)$ and $P_2 = \text{Ann} \left( QB \right)$, then $P_1 \neq P_2$ since $Q[G/A]/P_1 \cong Q$ and $Q[G/A]/P_2$ has dimension greater than 1. If $Ann U = Ann QB$, then $P_2 \subseteq P_1$, but this is impossible since each has cofinite dimension and hence are maximal. Therefore, $Ann U \not\subseteq Ann QB$. By [4], there is an $x \neq 0$ in the center of $Q[G/A]/Ann \left( U \right)$ with $x$ in $Ann \left( QB \right)/Ann \left( U \right)$. $x$ induces an isomorphism of $U/QB$ onto $QB$. But this is impossible. Therefore, we have $QA = QB$ and hence $A = B$. We now show $C(B) = B$. If not, take $x$ in $C(B) - B$ with $xB$ in $Z(G/B) \cap C(B)/B$. Then $D = \langle x, B \rangle$ is an abelian normal subgroup contradicting the maximality of $A = B$. Since $F$ is not algebraic over a finite field, there exists a monomorphism of $B$ into the multiplicative group of $F$. This defines an $F[B]$ module structure on $F$. We denote this module by $V$. Let $P = \text{Ann}_{F[B]} V$. $V = V \otimes_{F[B]} F[G]$ is an irreducible $F[G]$ module. This follows since if $g_1$ and $g_2$ are in different cosets of $B$, then $V_{g_1}$ and $V_{g_2}$ are not isomorphic as $F[B]$ modules since $C[B] = B$. This implies $g_1 P_{g_1}^{-1} \neq g_2 P_{g_2}^{-1}$. Annihilator of $V$ is $(\cap_{g \in G} g P^{-1} g)$. But this is zero by Bergman's theorem [1]. Hence $Ann \left( V \right)$ is a nonmaximal primitive ideal.

The proof can be simplified considerably if $F$ is a large field. More specifically, if the transcendence degree $F$ is larger than the rank of $G$, a
 theorem of Passman may be applied to produce lots of primitive ideals easily
[3].

REFERENCES


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