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ARTIN ROOT NUMBERS FOR REAL CHARACTERS

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ABSTRACT. For K and L number fields, χ a real-valued character on $\mathrm{Gal}(K/L)$, the Artin root number $W(\chi)$ is ± 1. We analyze the question of sign for χ a degree 2 character over $\mathbf{Q}$ induced from an abelian character on a quadratic extension.

Let $K/\mathbf{Q}$ be a finite Galois extension, and χ some character on $\mathrm{Gal}(K/\mathbf{Q})$. The Artin L-function [2] $L(s, \chi)$ satisfies a functional equation of form $\xi(s, \chi) = W(\chi)\xi(1 - s, \bar{\chi})$ where ξ is L multiplied by Γ factors and (constant)^S, and $W(\chi)$ is the Artin root number associated to χ. Iterating this functional equation gives $W(\chi)W(\bar{\chi}) = 1$. Further, if χ is real, then $W(\chi)^2 = 1$ or $W(\chi) = \pm 1$.

Until recently $W(\chi)$ was assumed always $+1$. The first examples to the contrary were given by Serre (unpublished) and Armitage [1] using a construction of Serre. The question of sign is of considerable interest: first because an L-function with $W(\chi) = -1$ gives a number field whose zeta function has a zero at $\frac{1}{2}$ (see [1]), and second because the question of sign is tied up with the existence of a normal basis for the ring of integers O_K over $\mathbf{Z}$ (see [4] and Fröhlich's Vancouver talk).

Here we consider the following set up: We start with k a quadratic extension of $\mathbf{Q}$ and ψ a primitive abelian character on k. Assume that K, the field defined by ψ via class field theory, is normal $/\mathbf{Q}$ and that the induced character $\chi = \psi^*$ on $\mathrm{Gal}(K/\mathbf{Q})$ is real valued. Then we prove:

(1) For k imaginary, $W(\chi) = +1$ always.

(2) For k real, tamely rarified $/\mathbf{Q}$, $W(\chi) = \pm 1$, and both occur infinitely often with given k.

This result is subsumed in a recent paper of Fröhlich [5] (where $\mathbf{Q}$ is replaced by an arbitrary number field). The proof here is less elegant but more elementary.

The main idea of the proof is to look at ψ restricted to $\mathbf{Z} \subset O_k$. This is a Dirichlet character on $\mathbf{Z}$ and in fact we have

PROPOSITION 1. *$\psi|\mathbf{Z}$ is a not necessarily primitive character cotrained either with the identity or with the Kronecker symbol (d/n), for d = discriminant of k.*

PROOF. Since $\chi = \psi^*$ is real, $L(s, \psi)$ has real coefficients as a Dirichlet series. That in turn says the product of the Euler factors above a single rational prime has real coefficients. Immediately we get

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$$\psi((p)) = \pm 1, 0 \quad \text{for } (d/p) = -1,$$
$$\psi(\underline{P}) = \pm 1, 0 \quad \text{for } (d/p) = 0,\ \underline{P}^2 = (p).$$

For $(d/p) = +1$,

$$(p) = \underline{P}\,\underline{P}^{\sigma} \quad \text{for } \sigma \in \mathrm{Gal}(k/Q).$$
$$(1 - (\underline{P})p^{-s})^{-1}(1 - (\underline{P}^{\sigma})p^{-s})^{-1}$$
$$= \left(1 - \left[(\underline{P}) + (\underline{P}^{\sigma})\right]p^{-s} + ((p))p^{-2s}\right)^{-1},$$

so $\psi(\underline{P}) + \psi(\underline{P}^{\sigma}) \in R$. Furthermore since K is normal $/\mathbf{Q}$, $\psi(\underline{P})$ and $\psi(\underline{P}^{\sigma})$ must be roots of unity of the same order. This implies

$$\psi(\underline{P}^{\sigma}) = \overline{\psi(\underline{P})}, \quad \text{and} \quad \psi((p)) = \psi(\underline{P})\psi(\underline{P}^{\sigma}) = 1.$$

We thus see that $\psi((p)) = +1$ or 0 unless $(d/p) = -1$. This means $\psi|\mathbf{Z}$ factors through the Artin symbol

$$\left[\frac{k/\mathbf{Q}}{p}\right],$$

which is the statement of the proposition.

Characters ψ such that $\psi|\mathbf{Z}$ is cotrained with the identity are precisely the classical ring class characters.

PROPOSITION 2. *Let ψ be a primitive ring class character on a quadratic field. Then $W(\psi) = 1$.*

PROOF. In [3] we explicitly evaluate the local factors in Tate's formula for $W(\psi)$. This is somewhat tedious but straightforward. Fröhlich's approach is similar. Stark (unpublished) has suggested a purely analytic proof based on the form of the functional equation for ring class characters.

THEOREM 1. *If k is imaginary quadratic $W(\chi) = +1$.*

PROOF. Since the Artin L-function is invariant under inducing characters, $W(\chi) = W(\psi)$. Next, since k is totally imaginary, ψ is unramified at ∞, so $\psi(-1) = +1$. By Proposition 1, $\psi|\mathbf{Z}$ is cotrained with the identity or with (d/n). For an imaginary quadratic field $(d/-1) = -1$, so ψ must be cotrained with the identity. Then by Proposition 2, $W(\psi) = +1$.

We now move on to the real quadratic case. Since Proposition 2 still applies, we need first to find characters ψ so that $\psi|\mathbf{Z}$ is cotrained with (d/n).

PROPOSITION 3. *If k is real, tamely ramified $/\mathbf{Q}$, there exist infinitely many characters ψ on k satisfying*

(a) *ψ is primitive,*

(b) *Ker ψ is invariant under conjugation,*

(c) *$\psi|\mathbf{Z}$ is cotrained with (d/n).*

PROOF. We explicitly construct ψ as a product $\psi_p\psi_d$, where ψ_p and ψ_d are numerical characters on O_k (not necessarily trivial on the ray class of the fundamental unit) with the following properties: ψ_p is defined modulo some

rational prime p and is trivial on integers; $\psi_d|\mathbf{Z}$ is (d/n). In fact for k tamely ramified we can take ψ_d to be (d/n) as defined on $O_k \bmod(\sqrt{d})$.

Let u be the fundamental unit. $\psi_d(u) = \pm 1$. If it is $+1$, then ψ_d itself extends to a ray class character. If $\psi_d(u) = -1$, then we look for a rational prime p and a character ψ_p such that $\psi_p(u) = -1$. $\psi_p\psi_d$ then will extend to a ray class character ψ.

The existence of the character ψ_p is an exercise in class field theory (see [3]). That there are infinitely many ψ follows since the existence proof finds infinitely many ψ_p. If ψ is any character with the desired properties, $\psi\psi_p\psi_{p'}$ is another.

As yet we do not have $W(\psi) = -1$. For that we use a very special case of a theorem of Langlands (see [6], or [4] for this form). Let θ, ν be ray class characters with conductors (finite parts) f_θ, f_ν. Let $M(\theta)$ be the number of real primes at which θ is ramified. If $(f_\theta, f_\nu) = 1$ and if $M(\theta) + M(\nu) = M(\theta\nu)$ (mod 4), then $W(\theta)W(\nu) = W(\theta\nu)\nu(f_\theta)\theta(f_\nu)$.

We now use class field theory to construct a ring class character ψ' such that

$$(f_\psi, f_{\psi'}) = 1, \quad \psi(f_{\psi'}) = -1, \quad \psi'(f_\psi) = 1.$$

Since ψ' is a ring class character it is either unramified or totally ramified at ∞. Further, the same is true for ψ by (b) of Proposition 3. Hence the condition at the infinite primes in Langlands' theorem is satisfied. $(f_\psi, f_{\psi'}) = 1$ takes care of the other condition.

Collecting information we have $W(\psi)W(\psi') = W(\psi\psi')\psi(f_{\psi'})\psi'(f_\psi)$ by Langlands; $W(\psi') = 1$ by Proposition 2; $\psi(f_{\psi'}) = -1$ and $\psi'(f_\psi) = 1$ by construction. Therefore

$$W(\psi) = -W(\psi\psi').$$

$\psi\psi'$ also satisfies (a), (b), and (c) of Proposition 3, so we have

THEOREM 2. *Let k be a real quadratic field, tamely ramified $/\mathbf{Q}$. Then there exist infinitely many ray class characters ψ on k such that*

(a) *ψ is primitive,*

(b) Ker ψ *is invariant under conjugation,*

(c) *$\psi|\mathbf{Z}$ is cotrained with (d/n),*

(d) $W(\psi) = -1$.

The same is true with (d) *replaced by* $W(\psi) = +1$.

Only condition (c) needs comment, for it shows (given (b)) that ψ^* is a real character. To see this we reverse the reasoning in Proposition 1.

$$\begin{aligned}
\psi^* \text{ is real} &\Leftrightarrow \psi^*([(K/\mathbf{Q})/\underline{P}]) \in R \text{ for all primes } \underline{P} \text{ of } K,\\
&\Leftrightarrow L(s, \psi^*) \text{ has real coefficients as a Dirichlet series,}\\
&\Leftrightarrow L(s, \psi) \text{ has real coefficients as a Dirichlet series,}\\
&\Leftrightarrow \psi((p)) = \pm 1, 0 \quad \text{for } (d/p) = -1,\\
& \psi((p)) = +1, 0 \quad \text{for } (d/p) = 1, 0,
\end{aligned}$$

and this last condition follows from (c). Consequently Theorem 2 is precisely the result asserted in the case of real k.

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