ARTIN ROOT NUMBERS FOR REAL CHARACTERS

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Abstract. For $K$ and $L$ number fields, $\chi$ a real-valued character on $\text{Gal}(K/L)$, the Artin root number $W(\chi)$ is $\pm 1$. We analyze the question of sign for $\chi$ a degree 2 character over $\mathbb{Q}$ induced from an abelian character on a quadratic extension.

Let $K/\mathbb{Q}$ be a finite Galois extension, and $\chi$ some character on $\text{Gal}(K/\mathbb{Q})$. The Artin $L$-function $L(s, \chi)$ satisfies a functional equation of form $\xi(s, \chi) = W(\chi)\xi(1 - s, \overline{\chi})$ where $\xi$ is $L$ multiplied by $\Gamma$ factors and (constant)$^8$, and $W(\chi)$ is the Artin root number associated to $\chi$. Iterating this functional equation gives $W(\chi)W(\chi) = 1$. Further, if $\chi$ is real, then $W(\chi)^2 = 1$ or $W(\chi) = \pm 1$.

Until recently $W(\chi)$ was assumed always $+1$. The first examples to the contrary were given by Serre (unpublished) and Armitage [1] using a construction of Serre. The question of sign is of considerable interest: first because an $L$-function with $W(\chi) = -1$ gives a number field whose zeta function has a zero at $\frac{1}{2}$ (see [1]), and second because the question of sign is tied up with the existence of a normal basis for the ring of integers $\mathcal{O}_K$ over $\mathbb{Z}$ (see [4] and Fröhlich's Vancouver talk).

Here we consider the following set up: We start with $k$ a quadratic extension of $\mathbb{Q}$ and $\psi$ a primitive abelian character on $k$. Assume that $K$, the field defined by $\psi$ via class field theory, is normal $/\mathbb{Q}$ and that the induced character $\chi = \psi^*$ on $\text{Gal}(K/\mathbb{Q})$ is real valued. Then we prove:

1. For $k$ imaginary, $W(\chi) = +1$ always.
2. For $k$ real, tamely rarified $/\mathbb{Q}$, $W(\chi) = \pm 1$, and both occur infinitely often with given $k$.

This result is subsumed in a recent paper of Fröhlich [5] (where $\mathbb{Q}$ is replaced by an arbitrary number field). The proof here is less elegant but more elementary.

The main idea of the proof is to look at $\psi$ restricted to $\mathbb{Z} \subset \mathcal{O}_k$. This is a Dirichlet character on $\mathbb{Z}$ and in fact we have

Proposition 1. $\psi|\mathbb{Z}$ is a not necessarily primitive character cotrained either with the identity or with the Kronecker symbol $(d/n)$, for $d = \text{discriminant of } k$.

Proof. Since $\chi = \psi^*$ is real, $L(s, \psi)$ has real coefficients as a Dirichlet series. That in turn says the product of the Euler factors above a single rational prime has real coefficients. Immediately we get
\[
\psi((p)) = \pm 1, 0 \quad \text{for} \quad (d/p) = -1,
\]
\[
\psi(P) = \pm 1, 0 \quad \text{for} \quad (d/p) = 0, P^2 = (p).
\]

For \((d/p) = +1,\)
\[
(P) = P^{\sigma} \quad \text{for} \quad \sigma \in \text{Gal}(k/Q).
\]
\[
(1 - (P)p^{-s})^{-1}(1 - (P^{\sigma})p^{-s})^{-1}
= \left(1 - \left[\left(\frac{P}{P^{\sigma}}\right) + (P^{\sigma})\right]p^{-s} + ((p))p^{-2s}\right)^{-1},
\]
so \(\psi(P) + \psi(P^{\sigma}) \in R.\) Furthermore since \(K\) is normal \(/Q,\) \(\psi(P)\) and \(\psi(P^{\sigma})\) must be roots of unity of the same order. This implies
\[
\psi(P^{\sigma}) = \overline{\psi(P)}, \quad \text{and} \quad \psi((p)) = \psi(P)\psi(P^{\sigma}) = 1.
\]
We thus see that \(\psi((p)) = +1\) or 0 unless \((d/p) = -1.\) This means \(\psi|Z\) factors through the Artin symbol
\[
\left[\frac{k/Q}{p}\right],
\]
which is the statement of the proposition.

Characters \(\psi\) such that \(\psi|Z\) is cotrained with the identity are precisely the classical ring class characters.

**Proposition 2.** Let \(\psi\) be a primitive ring class character on a quadratic field. Then \(W(\psi) = 1.\)

**Proof.** In [3] we explicitly evaluate the local factors in Tate's formula for \(W(\psi).\) This is somewhat tedious but straightforward. Fröhlich's approach is similar. Stark (unpublished) has suggested a purely analytic proof based on the form of the functional equation for ring class characters.

**Theorem 1.** If \(k\) is imaginary quadratic \(W(\chi) = +1.\)

**Proof.** Since the Artin \(L\)-function is invariant under inducing characters, \(W(\chi) = W(\psi).\) Next, since \(k\) is totally imaginary, \(\psi\) is unramified at \(\infty,\) so \(\psi(-1) = +1.\) By Proposition 1, \(\psi|Z\) is cotrained with the identity or with \((d/n).\) For an imaginary quadratic field \((d/-1) = -1,\) so \(\psi\) must be cotrained with the identity. Then by Proposition 2, \(W(\psi) = +1.\)

We now move on to the real quadratic case. Since Proposition 2 still applies, we need first to find characters \(\psi\) so that \(\psi|Z\) is cotrained with \((d/n).\)

**Proposition 3.** If \(k\) is real, tamely ramified \(/Q,\) there exist infinitely many characters \(\psi\) on \(k\) satisfying

\((a)\) \(\psi\) is primitive,
\((b)\) \(\text{Ker} \; \psi\) is invariant under conjugation,
\((c)\) \(\psi|Z\) is cotrained with \((d/n).\)

**Proof.** We explicitly construct \(\psi\) as a product \(\psi_p\psi_d,\) where \(\psi_p\) and \(\psi_d\) are numerical characters on \(O_k\) (not necessarily trivial on the ray class of the fundamental unit) with the following properties: \(\psi_p\) is defined modulo some
rational prime $p$ and is trivial on integers; $\psi_d|\mathbb{Z}$ is $(d/n)$. In fact for $k$ tamely ramified we can take $\psi_d$ to be $(d/n)$ as defined on $O_K \mod(\sqrt{d})$.

Let $u$ be the fundamental unit. $\psi_d(u) = \pm 1$. If it is $+1$, then $\psi_d$ itself extends to a ray class character. If $\psi_d(u) = -1$, then we look for a rational prime $p$ and a character $\psi_p$ such that $\psi_p(u) = -1$. $\psi_p \psi_d$ then will extend to a ray class character $\psi$.

The existence of the character $\psi_p$ is an exercise in class field theory (see [3]). That there are infinitely many $\psi$ follows since the existence proof finds infinitely many $\psi_p$. If $\psi$ is any character with the desired properties, $\psi \psi_p \psi_p'$ is another.

As yet we do not have $W(\psi) = -1$. For that we use a very special case of a theorem of Langlands (see [6], or [4] for this form). Let $\theta, \nu$ be ray class characters with conductors (finite parts) $f_\theta, f_\nu$. Let $M(\theta)$ be the number of real primes at which $\theta$ is ramified. If $(f_\theta, f_\nu) = 1$ and if $M(\theta) + M(\nu) = M(\theta \nu)$ (mod 4), then $W(\theta) W(\nu) = W(\theta \nu) \nu(f_\theta) \theta(f_\nu)$.

We now use class field theory to construct a ring class character $\psi'$ such that

$$(f_\psi, f_\psi') = 1, \quad \psi(f_\psi) = -1, \quad \psi'(f_\psi) = 1.$$

Since $\psi'$ is a ring class character it is either unramified or totally ramified at $\infty$. Further, the same is true for $\psi$ by (b) of Proposition 3. Hence the condition at the infinite primes in Langlands’ theorem is satisfied, $(f_\psi, f_\psi') = 1$ takes care of the other condition.

Collecting information we have $W(\psi) W(\psi') = W(\psi') \psi(f_\psi) \psi'(f_\psi)$ by Langlands; $W(\psi') = 1$ by Proposition 2; $\psi(f_\psi) = -1$ and $\psi'(f_\psi) = 1$ by construction. Therefore

$$W(\psi) = -W(\psi').$$

$\psi \psi'$ also satisfies (a), (b), and (c) of Proposition 3, so we have

**Theorem 2.** Let $k$ be a real quadratic field, tamely ramified $\mathbb{Q}$. Then there exist infinitely many ray class characters $\psi$ on $k$ such that

(a) $\psi$ is primitive,
(b) $\text{Ker } \psi$ is invariant under conjugation,
(c) $\psi|\mathbb{Z}$ is cotrained with $(d/n)$,
(d) $W(\psi) = -1$.

The same is true with (d) replaced by $W(\psi) = +1$.

Only condition (c) needs comment, for it shows (given (b)) that $\psi^*$ is a real character. To see this we reverse the reasoning in Proposition 1.

$\psi^*$ is real $\iff \psi^*((K/\mathbb{Q})/\mathbb{P}) \in \mathbb{R}$ for all primes $P$ of $K$,

$\iff L(s, \psi^*)$ has real coefficients as a Dirichlet series,

$\iff L(s, \psi)$ has real coefficients as a Dirichlet series,

$\iff \psi((p)) = \pm 1, 0 $ for $(d/p) = -1$,

$\psi((p)) = +1, 0 $ for $(d/p) = 1, 0$,

and this last condition follows from (c). Consequently Theorem 2 is precisely the result asserted in the case of real $k$.
References


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