RELATING GROUP TOPOLOGIES BY THEIR CONTINUOUS POINTS

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ABSTRACT. Let \( x \) be a point in a topological group \( G \), and for each integer \( n \), let \( (1/n)x \) be the set \( \{ y: ny = x \} \) in \( G \). Then I call \( x \) a continuous point if for positive integers \( n \), the subsets \( (1/n)x \) are nonvoid and eventually intersect each neighbourhood of the identity 0. I prove the following result and from it three corollaries. Let \( G \) be a divisible abelian group such that \( (1/n)0 = \{0\} \) for some integer \( n > 2 \). Suppose there are two group topologies \( \tau_1 \) and \( \tau_2 \) defined on \( G \) and that \( G \) is \( \tau_2 \)-locally compact and \( \sigma \)-compact, and define \( \omega_2 \) to be the outer measure derived from the Haar measure \( \mu_2 \) on \( (G, \tau_2) \). Also suppose that the ratio of the \( \tau_2 \)-measure of \( \{nx: x \in A\} \) to the \( \tau_2 \)-measure of \( A \), for any \( \tau_2 \)-Borel-measurable set \( A \) (the ratio is the same for any such \( A \) with finite measure), does not exceed 1. Then for each \( \tau_2 \)-Borel-measurable set \( A \) with nonvoid \( \tau_1 \)-interior, \( \mu_2(A) > \omega_2(W_1) \), \( W_1 \) being the subgroup of all points in \( G \) which are \( \tau_1 \)-continuous.

The study of compact group topologies for the real line gave rise to the rather interesting questions posed by D. N Hawley [1] and answered by me for \( \mathbb{R}^N \) [4]. I propose to present now a generalization of the proofs in [4], something which supplies the basis for the study of what I call the continuous points in a topological group (see [5]). This work forms part of a Ph.D. thesis submitted to La Trobe University in Melbourne, Australia, and was done under the supervision of Dr. Graham Elton.

DEFINITIONS. Let \( G \) be a group (I write my groups additively) and \( A \) a subset of \( G \). It is possible to define two kinds of "\( n \)th-multiples" of the set \( A \):

\[
\begin{align*}
nA &= \{x_1 + x_2 + \cdots + x_n: x_1, x_2, \ldots, x_n \in A\}, \\
\circ nA &= \{nx: x \in A\},
\end{align*}
\]

for \( n \) a positive integer, and

for \( n \) any integer.

An element \( x \) of \( G \) is divisible (in \( G \)) if for each positive integer \( n \) there is a \( y \) in \( G \) satisfying \( x = ny \). If every element of \( G \) is divisible in \( G \), then \( G \) is said to be divisible. To avoid excess of writing, I put \( (1/n)x = \{ y: ny = x \} \), and for \( A \) a subset of \( G \), \( \circ(1/n)A = \{ y: ny \in A \} \).

Now consider \( G \) to be a topological group. I call a divisible element \( x \) of \( G \)
a continuous point if the subsets \((1/n)x\), for positive integers \(n\), eventually intersect each neighbourhood of the identity. In other words, if \(A\) is a neighbourhood of the identity and \(x\) is a continuous point, there is a positive integer \(N\) such that for all \(n > N\), \((1/n)x \cap A \neq \emptyset\). I designate the collection of continuous points in \(G\) by \(W\), and if \(G\) is abelian, \(W\) is a subgroup.

Most of this work concerns groups which are divisible and abelian; these I call \(da\) groups for short. I am also concerned with the torsion-free property in that it involves this idea: \(G\) is uniquely \(n\)-th-rooted if \(y_1\) and \(y_2\) in \(G\) are such that \(ny_1 = ny_2\), then \(y_1 = y_2\) \((n\) is a positive integer). If \(G\) is an abelian group, then the uniquely \(n\)-th-rooted property is equivalent to \(G\)'s containing no points, except the identity 0, whose \(n\)-th-multiple is 0. (Note that if \(x\) in \(G\) is uniquely \(n\)-th-rooted, \((1/n)x\) contains at most one point, and I take \((1/n)x\) to be that point.)

First I want to show that \(\circ nA\) and \(\circ (1/n)A\) are Borel (-measurable) for a Borel set \(A\) and a positive integer \(n\).

**Lemma 1.** If \(G\) is a \(da\) \(\sigma\)-compact locally compact group, then, for each positive integer \(n\), the function \(f_n: x \rightarrow nx\), for all \(x\) in \(G\), is an open and continuous homomorphism of \(G\) onto \(G\).

**Proof.** That \(f_n\) is continuous follows simply from the definition of a topological group (see [3, p. 96, part A]), and the open property follows from (5.29) in [2, p. 42].

It is true then that, in any topological group, not only translates and inverses of Borel sets are again Borel, but also that the \(n\)-th-multiples of Borel sets are Borel when the group is \(\sigma\)-compact, locally compact and uniquely \(n\)-th-rooted.

To my main train of thought. I want to build up to the fact that for certain groups \(G\) with Haar measure \(\mu\), \(\mu(\circ nA) < \mu(A)\) for a positive integer \(n\) and for all \(A\) in \(\mathcal{M}\), the \(\sigma\)-algebra of all Borel sets. To do this define \(\mu^n\) by \(\mu^n(A) = \mu(\circ nA)\) for all \(A\) in \(\mathcal{M}\). If \(G\) is \(da\), uniquely \(n\)-th-rooted, locally compact and \(\sigma\)-compact, then \(\mu^n\) is a Haar measure on \(G\); for instance

\[
\mu^n(x + A) = \mu(\circ n(x + A)) = \mu(nx + \circ nA) = \mu(\circ nA) = \mu^n(A),
\]

for all \(x\) in \(G\) and \(A\) in \(\mathcal{M}\). But the Haar measure on \(G\) is essentially unique, and so there is a positive real \(c_n\) such that \(\mu^n = c_n \mu\). It can be shown that \(c_n\) is the product of integer powers of the prime factors of \(n\), but more important for this study, it can be shown that \(c_n\) is not dependent on the particular Haar measure chosen for the topology. If \(c_n < 1\), then \(\mu(\circ nA) = \mu^n(A) < \mu(A)\) for all \(A\) in \(\mathcal{M}\), and this would be the case if, for instance, \(G\) contains a compact open subgroup. To summarize these results:

**Lemma 2.** Let \(G\) be a \(da\), uniquely \(n\)-th-rooted, \(\sigma\)-compact, locally compact group with \(c_n < 1\) for some integer \(n > 2\). Then \(\mu(\circ nA) < \mu(A)\) for all Borel sets \(A\) and for a Haar measure \(\mu\).

Now to my main result.

**Theorem 3.** Let \(G\) be a \(da\), uniquely \(n\)-th-rooted group for some integer \(n > 2\). Suppose there are two group topologies \(\mathcal{G}_1\) and \(\mathcal{G}_2\) defined on \(G\), such that \((G,
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Theorem 3. Let $\mathcal{G}$ be a group, and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Proof. Let $\mathcal{G}$ be a group and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Theorem 4. Let $\mathcal{G}$ be a group, and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Proof. Let $\mathcal{G}$ be a group and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Theorem 5. Let $\mathcal{G}$ be a group, and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Proof. Let $\mathcal{G}$ be a group and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Theorem 6. Let $\mathcal{G}$ be a group, and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.

Proof. Let $\mathcal{G}$ be a group and let $\tau$ be a topology on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\tau$ is a Hausdorff topology. Then there is a topology $\sigma$ on $\mathcal{G}$ such that $\mathcal{G}$ is a Hausdorff space and $\sigma$ is a Hausdorff topology.
Proof. Putting $\mathcal{A}_1 = \mathcal{A}_2$ in Theorem 3 implies that every open set has measure at least the outer measure of the subgroup of continuous points. But by regularity, since a point has zero measure, there are sets open in $G$ with arbitrarily small measures.

Definition. Suppose there are two topologies $\mathcal{A}_1$ and $\mathcal{A}_2$ defined on some space $X$. Then $\mathcal{A}_2$ is Hawley with respect to $\mathcal{A}_1$ if, given any $\mathcal{A}_2$-Borel set, either it or its complement is dense in $(X, \mathcal{A}_1)$.

Corollary 6. Let $G$ be a da, uniquely nth-rooted group for some integer $n > 2$. Suppose there are two group topologies $\mathcal{A}_1$ and $\mathcal{A}_2$ defined on $G$, and $\mathcal{A}_2$ causes $G$ to be compact. Then $\mathcal{A}_2$ is Hawley with respect to $\mathcal{A}_1$ if the subgroup of $\mathcal{A}_1$-continuous points is not $\mathcal{A}_2$-negligible.

Proof. In any compact and connected group, a nonnegligible subgroup has outer measure 1. Now by applying Theorem 3 with $c_n = 1$ to our present group, it can be seen that every $\mathcal{A}_2$-Borel set with nonvoid $\mathcal{A}_1$-interior must have $\mathcal{A}_2$-measure 1. Thus the $\mathcal{A}_2$-measure of $G$ is two times what it should be if an $\mathcal{A}_2$-Borel set and its complement are both not dense in $(G, \mathcal{A}_1)$.

If $\mathcal{A}_2$ is Hawley with respect to $\mathcal{A}_1$ for two topologies $\mathcal{A}_1$ and $\mathcal{A}_2$ on some space $X$, then the only functions from $X$ to a Hausdorff space both $\mathcal{A}_1$-continuous and $\mathcal{A}_2$-Borel-measurable are the constant functions. This can be proved in exactly the same way as Theorem 4 in [4]. However, it is not possible to remove the "$\mathcal{A}_1$-continuous" and make it "$\mathcal{A}_2$-Borel-measurable" for if $(X, \mathcal{A}_1)$ and $(X, \mathcal{A}_2)$ are Hausdorff spaces and $X$ contains two distinct points $x$ and $y$, the map $f: X \to \{x, y\}$ defined by $f(x) = x$ and $f(z) = y$ if $z$ is in $\{x\}$, is $\mathcal{A}_1$ - and $\mathcal{A}_2$-Borel-measurable, but is not a constant function.

References

5. ———, Continuous points in topological groups (submitted).

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