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RELATING GROUP TOPOLOGIES BY THEIR CONTINUOUS POINTS

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ABSTRACT. Let x be a point in a topological group G, and for each integer n, let $(1/n)x$ be the set $\{y: ny = x\}$ in G. Then I call x a continuous point if for positive integers n, the subsets $(1/n)x$ are nonvoid and eventually intersect each neighbourhood of the identity 0. I prove the following result and from it three corollaries. Let G be a divisible abelian group such that $(1/n)0 = \{0\}$ for some integer $n \geqslant 2$. Suppose there are two group topologies $\mathcal{Q}_1$ and $\mathcal{Q}_2$ defined on G and that G is $\mathcal{Q}_2$-locally compact and σ-compact, and define ω_2 to be the outer measure derived from the Haar measure μ_2 on $(G, \mathcal{Q}_2)$. Also suppose that the ratio of the $\mathcal{Q}_2$-measure of $\{nx: x \in A\}$ to the $\mathcal{Q}_2$-measure of A, for any $\mathcal{Q}_2$-Borel-measurable set A (the ratio is the same for any such A with finite measure), does not exceed 1. Then for each $\mathcal{Q}_2$-Borel-measurable set A with nonvoid $\mathcal{Q}_1$-interior, $\mu_2(A) \geqslant \omega_2(W_1)$, W_1 being the subgroup of all points in G which are $\mathcal{Q}_1$-continuous.

The study of compact group topologies for the real line gave rise to the rather interesting questions posed by D. N Hawley [1] and answered by me for R^N [4]. I propose to present now a generalization of the proofs in [4], something which supplies the basis for the study of what I call the continuous points in a topological group (see [5]). This work forms part of a Ph.D. thesis submitted to La Trobe University in Melbourne, Australia, and was done under the supervision of Dr. Graham Elton.

DEFINITIONS. Let G be a group (I write my groups additively) and A a subset of G. It is possible to define two kinds of "nth-multiples" of the set A:

$$nA = \{x_1 + x_2 + \cdots + x_n: x_1, x_2, , \ldots, x_n \in A\},$$

for n a positive integer, and

$$^\circ nA = \{nx: x \in A\},$$

for n any integer.

An element x of G is *divisible* (in G) if for each positive integer n there is a y in G satisfying $x = ny$. If every element of G is divisible in G, then G is said to be divisible. To avoid excess of writing, I put $(1/n)x = \{y: ny = x\}$, and for A a subset of G, $^\circ(1/n)A = \{y: ny \in A\}$.

Now consider G to be a topological group. I call a divisible element x of G

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a *continuous point* if the subsets $(1/n)x$, for positive integers n, eventually intersect each neighbourhood of the identity. In other words, if A is a neighbourhood of the identity and x is a continuous point, there is a positive integer N such that for all $n > N$, $(1/n)x \cap A \neq \emptyset$. I designate the collection of continuous points in G by W, and if G is abelian, W is a subgroup.

Most of this work concerns groups which are divisible and abelian; these I call *da* groups for short. I am also concerned with the torsion-free property in that it involves this idea: G is *uniquely nth-rooted* if y_1 and y_2 in G are such that $ny_1 = ny_2$, then $y_1 = y_2$ (n is a positive integer). If G is an abelian group, then the uniquely nth-rooted property is equivalent to G's containing no points, except the identity 0, whose nth-multiple is 0. (Note that if x in G is uniquely nth-rooted, $(1/n)x$ contains at most one point, and I take $(1/n)x$ to be that point.)

First I want to show that $°nA$ and $°(1/n)A$ are Borel (-measurable) for a Borel set A and a positive integer n.

LEMMA 1. *If G is a da σ-compact locally compact group, then, for each positive integer n, the function f_n: $x \to nx$, for all x in G, is an open and continuous homomorphism of G onto G.*

PROOF. That f_n is continuous follows simply from the definition of a topological group (see [3, p. 96, part A]), and the open property follows from (5.29) in [2, p. 42].

It is true then that, in any topological group, not only translates and inverses of Borel sets are again Borel, but also that the nth-multiples of Borel sets are Borel when the group is σ-compact, locally compact and uniquely nth-rooted.

To my main train of thought. I want to build up to the fact that for certain groups G with Haar measure μ, $\mu(°nA) \leqslant \mu(A)$ for a positive integer n and for all A in $\mathfrak{M}$, the σ-algebra of all Borel sets. To do this define μ^n by $\mu^n(A) = \mu(°nA)$ for all A in $\mathfrak{M}$. If G is da, uniquely nth-rooted, locally compact and σ-compact, then μ^n is a Haar measure on G; for instance

$$\mu^n(x + A) = \mu(°n(x + A)) = \mu(nx + °nA) = \mu(°nA) = \mu^n(A),$$

for all x in G and A in $\mathfrak{M}$. But the Haar measure on G is essentially unique, and so there is a positive real c_n such that $\mu^n = c_n\mu$. It can be shown that c_n is the product of integer powers of the prime factors of n, but more important for this study, it can be shown that c_n is not dependent on the particular Haar measure chosen for the topology. If $c_n \leqslant 1$, then $\mu(°nA) = \mu^n(A) \leqslant \mu(A)$ for all A in $\mathfrak{M}$, and this would be the case if, for instance, G contains a compact open subgroup. To summarize these results:

LEMMA 2. *Let G be a da, uniquely nth-rooted, σ-compact, locally compact group with $c_n \leqslant 1$ for some integer $n \geqslant 2$. Then $\mu(°nA) \leqslant \mu(A)$ for all Borel sets A and for a Haar measure μ.*

Now to my main result.

THEOREM 3. *Let G be a da, uniquely nth-rooted group for some integer $n \geqslant 2$. Suppose there are two group topologies $\mathfrak{A}_1$ and $\mathfrak{A}_2$ defined on G, such that (G,*

$\mathcal{Q}_2$) is locally compact, σ-compact and for it $c_n \leqslant 1$, and define ω_2 to be the outer measure derived from the Haar measure μ_2 on $(G, \mathcal{Q}_2)$. Then for any $\mathcal{Q}_2$-Borel set A with $\mathcal{Q}_1$-interior containing 0,

$$\omega_2(W_1) \leqslant \mu_2\left[\bigcup_{v=0}^{\infty} {}^\circ n^v\left(\bigcap_{m=0}^{\infty} {}^\circ n^m A\right)\right] = \mu_2\left(\bigcap_{m=0}^{\infty} {}^\circ n^m A\right) \leqslant \mu_2(A),$$

where W_1 is the subgroup of all the points in G which are $\mathcal{Q}_1$-continuous.

PROOF. Let A be such an $\mathcal{Q}_2$-Borel set which contains the $\mathcal{Q}_1$-open neighbourhood B of 0. Then let $K = \bigcap_{m=0}^{\infty} {}^\circ n^m A$, and $L = \bigcup_{v=0}^{\infty} {}^\circ n^v K$; from Lemma 1 it follows that K and, hence, L are $\mathcal{Q}_2$-Borel.

As each x in W_1 is an $\mathcal{Q}_1$-continuous point, the sets $(1/u)x$, for positive integers u, eventually intersect B and, hence, they eventually intersect A. That is: there is a positive integer U such that for any $u > U$, $(1/u)x \cap A \neq \varnothing$. Take q an integer such that $n^q > U$. Then for all $m \geqslant 0$, $n^m n^q > U$ and $(1/n^m n^q)x \cap A \neq \varnothing$. Now for any positive integers a and c,

$$\begin{aligned}(1/a)[(1/c)x] &= {}^\circ(1/a)\{y \in G: cy = x\}\\ &= \{z \in G: az = y \text{ and } cy = x \text{ for some } y \in G\}\\ &= \{z \in G: acz = x\} = (1/ac)x.\end{aligned}$$

Hence $(1/n^m)[(1/n^q)x] \cap A \neq \varnothing$ for all $m \geqslant 0$. This and the fact that G is uniquely n^qth-rooted put $(1/n^q)x$ in ${}^\circ n^m A$, for all $m \geqslant 0$, and thus in K. So x is in ${}^\circ n^q K \subseteq L$, and $W_1 \subseteq L$.

I want to show that $\{{}^\circ n^v K\}_{v=0}^{\infty}$ is an ordered chain of subsets, that is, ${}^\circ n^v K \subseteq {}^\circ n^{v+1} K$ for any positive integer v. If x is in ${}^\circ n^v K$, then $(1/n^v)x$ is in K and $(1/n^v)x$ is in ${}^\circ n^m A$ for all $m \geqslant 0$. Hence, $(1/n^v)x$ is an element of ${}^\circ n({}^\circ n^m A) = {}^\circ n^{m+1} A$ for all $m \geqslant 0$, and taking nth-roots, $(1/n)[(1/n^v)x]$ is an element of ${}^\circ n^m A$ for all $m \geqslant 0$. So $(1/n^{v+1})x$ is in $\bigcap_{m=0}^{\infty} {}^\circ n^m A = K$ and x in ${}^\circ n^{v+1} K$, making ${}^\circ n^v K \subseteq {}^\circ n^{v+1} K$.

The fact that ${}^\circ n^v K \subseteq {}^\circ n^{v+1} K$ for all $v \geqslant 0$ means that

$$\mu_2\left(\bigcup_{v=0}^{\infty} {}^\circ n^v K\right) = \lim_{v \to \infty} \mu_2({}^\circ n^v K),$$

which is less than $\mu_2(K)$ by Lemma 2. Combining this with the facts that $W_1 \subseteq L$, $K \subseteq L$, and $A \supseteq K$, gives $\omega_2(W_1) \leqslant \mu_2(L) = \mu_2(K) \leqslant \mu_2(A)$, the required result.

There are three corollaries from this result, the second two of which form the basis of further work in this field (see [5]). The definition stated below arises from a generalization of the condition Hawley was interested in for the reals, and is used in Corollary 6.

COROLLARY 4. *Let G, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be as in Theorem 3, except that for the second topology $c_n < 1$. Then W_1 is $\mathcal{Q}_2$-negligible if there is an $\mathcal{Q}_2$-Borel set with finite $\mathcal{Q}_2$-measure and having a nonvoid $\mathcal{Q}_1$-interior.*

COROLLARY 5. *Let G be a da, uniquely nth-rooted, nondiscrete topological group for some integer $n \geqslant 2$, and which is locally compact, σ-compact and has $c_n \leqslant 1$. Then the subgroup of continuous points in G is negligible.*

PROOF. Putting $\mathcal{C}_1 = \mathcal{C}_2$ in Theorem 3 implies that every open set has measure at least the outer measure of the subgroup of continuous points. But by regularity, since a point has zero measure, there are sets open in G with arbitrarily small measures.

DEFINITION. Suppose there are two topologies $\mathcal{C}_1$ and $\mathcal{C}_2$ defined on some space X. Then $\mathcal{C}_2$ is *Hawley* with respect to $\mathcal{C}_1$ if, given any $\mathcal{C}_2$-Borel set, either it or its complement is dense in $(X, \mathcal{C}_1)$.

COROLLARY 6. *Let G be a da, uniquely nth-rooted group for some integer $n \geqslant 2$. Suppose there are two group topologies $\mathcal{C}_1$ and $\mathcal{C}_2$ defined on G, and $\mathcal{C}_2$ causes G to be compact. Then $\mathcal{C}_2$ is Hawley with respect to $\mathcal{C}_1$ if the subgroup of $\mathcal{C}_1$-continuous points is not $\mathcal{C}_2$-negligible.*

PROOF. In any compact and connected group, a nonnegligible subgroup has outer measure 1. Now by applying Theorem 3 with $c_n = 1$ to our present group, it can be seen that every $\mathcal{C}_2$-Borel set with nonvoid $\mathcal{C}_1$-interior must have $\mathcal{C}_2$-measure 1. Thus the $\mathcal{C}_2$-measure of G is two times what it should be if an $\mathcal{C}_2$-Borel set and its complement are both not dense in $(G, \mathcal{C}_1)$.

If $\mathcal{C}_2$ is Hawley with respect to $\mathcal{C}_1$ for two topologies $\mathcal{C}_1$ and $\mathcal{C}_2$ on some space X, then the only functions from X to a Hausdorff space both $\mathcal{C}_1$-continuous and $\mathcal{C}_2$-Borel-measurable are the constant functions. This can be proved in exactly the same way as Theorem 4 in [4]. However, it is not possible to remove the "$\mathcal{C}_1$-continuous" and make it "$\mathcal{C}_1$-Borel-measurable", for if $(X, \mathcal{C}_1)$ and $(X, \mathcal{C}_2)$ are Hausdorff spaces and X contains two distinct points x and y, the map f: $X \to \{x, y\}$ defined by $f(x) = x$ and $f(z) = y$ if z is in $\{x\}'$, is $\mathcal{C}_1$ - and $\mathcal{C}_2$-Borel-measurable, but is not a constant function.

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