PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 57, Number 1, May 1976

ONE DIMENSIONAL PERTURBATIONS OF COMPACT OPERATORS

HARKRISHAN VASUDEVA

ABSTRACT. Let A be a compact selfadjoint operator acting on a Hilbert space H. P denotes a one dimensional projection also acting on H. It is shown that the eigenvalues of A and $A + tP$ $(t > 0)$ interlace on the real axis. A converse of this result is also proved.

The purpose of this note is to study the change which results in the spectrum of a compact selfadjoint operator acting on a Hilbert space H by the addition to it of a positive multiple of a one dimensional projection.

We begin by stating a theorem of Hochstadt [1] and giving a variant of his proof based on [3] depending on a study of the resolvents of the operators involved, and our methods lead us naturally to a converse to this result.

Let A be a compact selfadjoint operator acting on a Hilbert space H. P denotes a one dimensional projection in the direction of a normalised element x of H: then $Py = (y,x)x$ for every y in H. B is the operator $A + tP$, $t > 0$.

THEOREM 1. *Suppose that the null space of A is empty. Between every pair of distinct, successive eigenvalues $(\lambda_i, \lambda_{i+1})$ there is precisely one eigenvalue of B in one of the intervals $(\lambda_i, \lambda_{i+1}]$, $[\lambda_i, \lambda_{i+1})$ or $(\lambda_i, \lambda_{i+1})$.*

PROOF. We can associate with A [2] a complete orthonormal set $\{\phi_i\}$ such that each ϕ_i is an eigenvalue of A so that $A\phi_i = \lambda_i\phi_i$ and $\lambda_i \neq 0$ for all i, by hypothesis. We introduce the resolvents of A and B:

$$R_\zeta = (A - \zeta I)^{-1}, \quad R'_\zeta = (B - \zeta I)^{-1}.$$

If u is an arbitrary element of H, then $R'_\zeta u = v$ (say). Now

$$u = (B - \zeta I)v = (A - \zeta I)v + tPv = (A - \zeta I)v + t(v,x)x,$$

and applying the resolvent R_ζ, we obtain

$$R_\zeta u = v + t(v,x)R_\zeta x$$

and $v = R'_\zeta u = R_\zeta u - t(v,x)R_\zeta x$.

Taking the inner product with x,

$$(v,x) = (R_\zeta u, x) - t(v,x)(R_\zeta x, x).$$

Solving for (v, x), we obtain

Received by the editors January 13, 1975 and, in revised form, March 4, 1975 and April 18, 1975.

AMS (MOS) subject classifications (1970). Primary 47A55, 47B05; Secondary 34B25.

$$(v, x) = (R_\zeta u, x)/(1 + t(R_\zeta x, x)).$$

Substituting for (v, x), we obtain

$$R'_\zeta u = R_\zeta u - \frac{t(R_\zeta u, x)}{1 + t(R_\zeta x, x)} R_\zeta x.$$

Moreover,

$$(R_\zeta x, x) = \left((A - \zeta I)^{-1} \sum_{j=1}^{\infty} (x, \phi_j)\phi_j, \sum_{k=1}^{\infty} (x, \phi_k)\phi_k\right)$$

$$= \sum_{j=1}^{\infty} |(x, \phi_j)|^2((A - \zeta I)^{-1}\phi_j, \phi_j) = \sum_{j=1}^{\infty} \frac{|(x, \phi_j)|^2}{\lambda_j - \zeta}.$$

Hence

$$R'_\zeta u = R_\zeta u - (t(R_\zeta u, x)/\phi(\zeta))R_\zeta x$$

where

$$\phi(\zeta) = 1 + t(R_\zeta x, x) = 1 + t \sum_{j=1}^{\infty} \frac{|(x, \phi_j)|^2}{\lambda_j - \zeta}.$$

Now any zero of $\phi(\zeta)$ is a pole of R'_ζ, hence an eigenvalue μ of B. Since the eigenvalues of A are not eigenvalues of B, it follows that the zeros of $\phi(\zeta)$ are the only poles of R'_ζ.

We shall assume, for the moment, that all the eigenvalues of A are simple and that (x, ϕ_j) vanishes for no j. This hypothesis ensures that $\phi(\zeta)$ has exactly one zero in every interval of the form $(\lambda_i, \lambda_{i+1})$.

In the general case when x is orthogonal to one or more ϕ_j $(j = 1, 2, \ldots)$ we perturb P to obtain P' so that P' is the projection in the direction of x', where x' is a normalised element of the Hilbert space, and (x', ϕ_j) vanishes for no j. If $\{\mu'_j\}$ are the eigenvalues of $B' = A + tP'$, then exactly one μ'_j lies in $(\lambda_j, \lambda_{j+1})$. In the limit as $B' \to B$, we have that μ_j belongs either to $[\lambda_j, \lambda_{j+1})$ or $(\lambda_j, \lambda_{j+1}]$.

THEOREM 2. *Let $\{\lambda_i\}$ and $\{\mu_i\}$ be two distinct monotone sequences of real numbers, each having zero as the only limit point. Further assume that μ_j belongs to $(\lambda_j, \lambda_{j+1})$ for each j and $\sum_{k=1}^{\infty} (\mu_k - \lambda_k)$ converges. Let A be a compact selfadjoint operator on a Hilbert space H having the λ_i $(i = 1, 2, \ldots)$ for eigenvalues. Then there exists a normalised element x and a corresponding one dimensional projection P such that for an appropriate $t > 0$ the operator $B = A + tP$ has the eigenvalues μ_i $(i = 1, 2, \ldots)$.*

PROOF. We consider the function

$$\phi(\zeta) = \prod_{k=1}^{\infty} \frac{\zeta - \mu_k}{\zeta - \lambda_k} = \prod_{k=1}^{\infty} \left(1 + \frac{\lambda_k - \mu_k}{\zeta - \lambda_k}\right).$$

Since

$$\sum |(\lambda_k - \mu_k)/(\zeta - \lambda_k)| = \sum (\mu_k - \lambda_k)/|\zeta - \lambda_k|$$

it follows, in view of the fact that $\Sigma (\mu_k - \lambda_k)$ is convergent, that

$$\prod_{k=1}^{\infty} (1 + (\lambda_k - \mu_k)/(\zeta - \lambda_k))$$

converges absolutely and uniformly on compact subsets which avoid the eigenvalues of A. Hence $\phi(\zeta)$ represents an analytic function in the region not containing $\{\lambda_i\}$.

Between any two successive poles λ_k, λ_{k+1} there is precisely one zero μ_k and that zero is simple. Now if the residue of $\phi(\zeta)$ at those poles were of different sign, the function, near the ends of the interval $(\lambda_k, \lambda_{k+1})$ would be large in absolute value and would have the same sign near the poles. Thus there would be an even number of zeros in between, multiplicity counted. Since this is not the case, the residues have the same sign at every pole. It is fairly easy to show that the function $\phi(\zeta)$ has negative residue at $\zeta = \lambda_1$. We may therefore write the function as

$$\phi(\zeta) = 1 + \sum_{k=1}^{\infty} \frac{m_k}{\lambda_k - \zeta}$$

where the masses m_k are positive. It is also easy to see that

$$\lim_{x \to \infty} x(1 - \phi(x)) = \sum_{k=1}^{\infty} m_k$$

and this limit can also be computed from the product representation of $\phi(\zeta)$; we find $\sum m_k = \sum (\mu_k - \lambda_k)$.

Call this quantity t and write $\phi(\zeta) = 1 + t \sum_k (m'_k)/(\lambda_k - \zeta)$. Evidently $\sum m'_k = 1$ and so if we choose a vector $x = \sum a_k \phi_k$ where ϕ_k are the normalised eigenvectors of A and a_k any solution of the equation $a_k^2 = m'_k$, then x is normalised. The operator $B = A + tP$ where P is the projection associated with x is then associated with $\phi(\zeta)$ and has the prescribed μ_k for its eigenvalues.

ACKNOWLEDGEMENT. The author would like to thank the Science Research Council of Great Britain for enabling him to visit the University of Sheffield while this work was done.

REFERENCES

1. H. Hochstadt, *One dimensional perturbations of compact operators*, Proc. Amer. Math. Soc. **37** (1973), 465–467. MR **46** #9779.

2. F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, 2nd ed., Akad. Kiadó, Budapest, 1952; English transl., Ungar, New York, 1955, p. 234. MR **14**, 286; **15**, 132; **17**, 175.

3. H. L. Vasudeva, *Monotone matrix functions*, Dissertation, University of California, Irvine, 1970.

DEPARTMENT OF MATHEMATICS, PUNJAB UNIVERSITY, CHANDIGARH 14, INDIA