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A GENERALIZATION OF A THEOREM BY BOCHNER

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ABSTRACT. A theorem of Bochner states that if μ is a complex Borel measure on the n-dimensional torus $\mathbf{T}^n$ with Fourier-coefficients that vanish outside a proper cone in $\mathbf{Z}^n$, then μ is absolutely continuous with respect to Haar measure on $\mathbf{T}^n$. This result is generalized to a C^*-algebra setting using the concept of spectral subspaces for an n-parameter group of automorphisms and its dual group, in the case where the cone is the positive "octant".

Introduction. The classical F. and M. Riesz theorems state that a complex Borel measure on the unit circle $\mathbf{T}$ which is analytic, i.e. its negative Fourier-coefficients all vanish, is either identically zero or has the same null-sets as Haar measure.

In [2], S. Bochner showed that a measure on $\mathbf{T}^n$ with Fourier-coefficients that vanish outside a proper cone in $\mathbf{Z}^n$ is absolutely continuous with respect to Haar measure on $\mathbf{T}^n$.

In [4], F. Forelli regarded an action of the real line $\mathbf{R}$ as a topological transformation group on the locally compact Hausdorff space S. He defined a measure λ to be *analytic* if its spectrum $\mathrm{sp}(\lambda)$ with respect to the action was nonnegative, and to be *quasi-invariant* if its collection of null-sets was carried into itself by the action of $\mathbf{R}$ on S. As a generalization of the F. and M. Riesz theorems he obtained the result that analytic measures are quasi-invariant. The notion of quasi-invariance was extended to the class of positive linear functionals on a C^*-algebra acted upon by a group of $*$-automorphisms in [5].

As a noncommutative generalization of Forelli's result, W. Arveson proved [1] that if a linear functional ϕ on a C^*-algebra had nonnegative spectrum with respect to a one-parameter group of $*$-automorphisms, the variation $|\phi|$ was quasi-invariant.

Here we show that Arveson's result can be used to prove quasi-invariance of functionals with spectrum in $[0, \infty)^n$, in the case of an n-parameter group action on a C^*-algebra. As a corollary we obtain that an analytic measure on $\mathbf{T}^n$ is either identically zero or has the same null-sets as Haar measure.

(For a short proof of Bochner's theorem in two dimensions, the reader may consult [8, 8.2.5].)

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Notation. In the following, A denotes a C^*-algebra, A' its dual space. By a representation (α, A) of $\mathbf{R}^n$ we mean a homomorphism from $\mathbf{R}^n$ into the group of $*$-automorphisms of A such that for every x in A

$$\|\alpha_{(t_1,\ldots,t_n)}x - x\| \to 0 \quad \text{as } (t_1,\ldots,t_n) \to (0,\ldots,0).$$

Let $\alpha(f)x$ with $f \in L^1(\mathbf{R}^n)$, $x \in A$ denote the vector-valued integral

$$\alpha(f)x = \int_{\mathbf{R}^n} f(t_1,\ldots,t_n)\alpha_{(t_1,\ldots,t_n)}x\,dt_1 \cdots dt_n.$$

Take $E \subseteq \mathbf{R}^n$. Define

$$R^\alpha(E) = [\{\alpha(f)x | \operatorname{supp}\hat{f} \subset E, x \in A\}],$$

[ ] denoting closed linear span,

$$\hat{f}(s_1,\ldots,s_n) = \int_{\mathbf{R}^n} f(t_1,\ldots,t_n)\exp\left(i \sum s_k t_k\right) dt_1,\ldots, dt_n.$$

Define for a closed set $E \subseteq \mathbf{R}^n$

$$M^\alpha(E) = \{x \in A | \alpha(f)x = 0 \ \forall f \in I_0(E)\}$$

where $I_0(E) = \{f \in L^1(\mathbf{R}^n) | \hat{f} \text{ vanishes on a neighbourhood of } E\}$.

Let ϕ be a positive linear functional on A. Let $(\pi_\phi, H_\phi, \xi_\phi)$ denote the representation obtained from ϕ using the Gelfand-Naimark-Segal construction [3, 2.4.4].

With (α, A) a representation of $\mathbf{R}^n$, the transposed action

$$(\alpha'_t\phi)(x) = \phi(\alpha_{-t}x) \quad \forall t \in \mathbf{R}^n \ \forall x \in A \ \forall \phi \in A'$$

defines a group of isometries of A' which takes positive functionals into positive functionals and satisfies that for t in $\mathbf{R}^n$,

$$|\alpha'_t\phi(x) - \phi(x)| \to 0 \quad \text{as } t \to 0 \ \forall x \in A.$$

We say that ϕ in A' is *continuous* if in fact $\|\alpha'_t\phi - \phi\| \to 0$ as $t \to 0$. We use $|\phi|$ to denote the *absolute value* or *variation* of ϕ, obtained by polar decomposition [3, 12.2].

We call ϕ *quasi-invariant* if the representations $\pi_{|\phi|}$ and $\pi_{\alpha'_t|\phi|}$ are quasi-equivalent [3, 5.3].

When T is a locally compact space, $C_0(T)$ denotes the continuous functions vanishing at infinity, and $M(T)$ the dual space of $C_0(T)$.

The main theorem.

THEOREM. *Let (α, A) be a representation of $\mathbf{R}^n$. Given a coordinate system, let $V = [0, \infty)^n$, and let CV denote the complement of V. If ϕ is a bounded linear functional on A which for some $s = (s_1,\ldots,s_n)$ in $\mathbf{R}^n$ satisfies the equivalent conditions*

(i) *ϕ annihilates $R^\alpha(s - CV)$,*

(ii) *$\phi \in M^{\alpha'}(-s + V)$,*

then $|\phi|$ is quasi-invariant and continuous.

PROOF. That (i) and (ii) are equivalent is proved in [7, Proposition 2.3.4(iii)] as a consequence of the duality relation $(\alpha_t'\phi)(x) = \phi(\alpha_{-t}x)$. Condition (i) entails that ϕ annihilates each $R^{\alpha^i}(s_i, \infty)$, $\alpha_{t_i}^i$ denoting the one-parameter subgroup $\alpha_{(0,\ldots,0,t_i,0,\ldots,0)}$, since

$$s - CV = (s_1, \ldots, s_n) - CV = \bigcup_{i=1}^{n} (s_i, \infty) \underset{j \neq i}{\times} \mathbf{R}_j$$

and so by [7, Proposition 2.3.4(ii) and Lemma 2.4.8(iv)], $R^\alpha(s - CV)$ is the closed linear span of spaces $R^{\alpha^i}(s_i, \infty)$. For brevity, let (π, H, ξ) denote the representation obtained from $|\phi|$ using the Gelfand-Naimark-Segal construction. By the generalization to C^*-algebras of the F. and M. Riesz theorems [1, Theorem 5.3], $|\phi|$ is then covariant with respect to each subgroup α^i, i.e. there exist strongly continuous one-parameter unitary groups u^i in $B(H)$ such that

$$\pi(\alpha_{t_i}^i(x)) = u_{t_i}^i \pi(x) u_{-t_i}^i \quad \forall t_i \in \mathbf{R}\ \forall x \in A.$$

It follows that

$$|\phi|(\alpha_{(t_1,\ldots,t_n)}x) = (\pi(x)u_{-t_n}^n \cdots u_{-t_1}^1 \xi | u_{-t_n}^n \cdots u_{-t_1}^1 \xi).$$

This shows that π and $\pi_{\alpha_t'|\phi|}$ are unitarily equivalent; thus $|\phi|$ is quasi-invariant. Furthermore

$$\begin{aligned}(\alpha_{t_i}^i|\phi| - |\phi|)(x) &= (\pi(x)u_{t_i}^i \xi | u_{t_i}^i \xi) - (\pi(x)\xi|\xi)\\ &= (\pi(x)(u_{t_i}^i \xi - \xi)|u_{t_i}^i \xi) + (\pi(x)\xi|u_{t_i}^i \xi - \xi)\end{aligned}$$

implies that

$$\|(\alpha_{t_i}^i)'|\phi| - |\phi|\,\| \leqslant 2\|u_{t_i}^i \xi - \xi\|;$$

thus

$$\|(\alpha_t)'|\phi| - |\phi|\,\| \to 0 \quad \text{as } t = (t_1, \ldots, t_n) \to 0.$$

REMARK. If the cyclic vector ξ is also separating for the weak closure $\pi(A)''$ of $\pi(A)$ on H, then quasi-invariance of $|\phi|$ implies covariance, i.e. the existence of a strongly continuous unitary group in $B(H)$ such that for every x in A and $(t_1, \ldots, t_n)$ in $\mathbf{R}^n$

$$\pi(\alpha_{(t_1,\ldots,t_n)}(x)) = u_{(t_1,\ldots,t_n)}\pi(x)u_{(-t_1,\ldots,-t_n)}.$$

To see this, note that quasi-invariance means that the automorphism group on A carries over to $\pi(A)''$. By [6, Corollary 3.6] a weakly continuous group of automorphisms on a von Neumann algebra has a strongly continuous unitary implementation whenever the group is locally compact and the algebra has a cyclic and separating vector.

It follows that if A is commutative, quasi-invariance and covariance are identical notions.

Thus it is an immediate consequence of the Theorem above that we have the following

COROLLARY. *Let T be a locally compact Hausdorff space, $(\alpha, C_0(T))$ a representation of $\mathbf{R}^n$. Let μ in $M(T)$ for some $s = (s_1, \ldots, s_n)$ in $\mathbf{R}^n$ satisfy the equivalent conditions ($V = [0, \infty)^n$ in some coordinate system):*
(i) *μ annihilates $R^\alpha(s - CV)$,*
(ii) *$\mu \in M^{\alpha'}(-s + V)$.*
Then the total variation $|\mu|$ of μ is covariant.

To relate this to the classical setting for the theorem of Bochner, let $\mathbf{T}^n$ denote the n-dimensional torus. Define an n-parameter group of $*$-automorphisms $\alpha_{(t_1, \ldots, t_n)}$ of $C(\mathbf{T}^n)$ by

$$(\alpha_{(t_1, \ldots, t_n)} f)(e^{ix_1}, \ldots, e^{ix_n}) = f^{i(x_1 - t_1)}, \ldots, e^{i(x_n - t_n)}).$$

Let μ be a complex Borel measure on $\mathbf{T}^n$. It is seen using the density of trigonometric polynomials in $C(\mathbf{T}^n)$ that μ annihilates all f in $R^\alpha((s_1, \ldots, s_n) - CV)$ if and only if $\hat{\mu}$ has support in $(-s_1, \ldots, -s_n) + V$. Thus $\operatorname{supp} \hat{\mu} \subset V$ implies that μ annihilates $R^\alpha(-CV)$ and we have that $|\mu|$ is covariant. So $|\mu|$ is either identically 0 or mutually equivalent to Haar measure on $\mathbf{T}^n$.

REFERENCES

1. W. Arveson, *On groups of automorphisms of operator algebras*, J. Functional Analysis **15** (1974), 217–243.

2. S. Bochner, *Boundary values of analytic functions in several variables and of almost periodic functions*, Ann. of Math. (2) **45** (1944), 708–722. MR **6**, 124.

3. J. Dixmier, *Les C*-algèbres et leurs représentations*, Cahiers Scientifiques, fasc. 29, Gauthier-Villars, Paris, 1964. MR **29** #485.

4. F. Forelli, *Analytic and quasi-invariant measures*, Acta Math. **118** (1967), 33–59. MR **35** #667.

5. A. Guichardet et D. Kastler, *Désintégration des états quasi-invariants des C*-algébres*, J. Math. Pures Appl. (9) **49** (1970), 349–380. MR **43** #2524.

6. U. Haagerup, *The standard form of von Neumann algebras*, Math. Scand. **37** (1975), 271–283.

7. D. Olesen, *On spectral subspaces and their applications to automorphism groups*, (Proc. Meeting "Algebre C* e loro applicazioni fisica teorica," Rome, March 1975) Luminy Lectures 1973-74–Marseille CNRS Preprint 74/P602 (to appear).

8. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Appl. Math., no. 12, Interscience, New York, 1962. MR **27** #2808.

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