JOIN-PRINCIPAL ELEMENT LATTICES

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Abstract. Let $(\mathcal{L}, M)$ be a local Noether lattice. If the maximal element $M$ is meet principal, it is well known and easily seen that every element of $\mathcal{L}$ is meet principal. In this note, we obtain the corresponding result for $M$ join-principal. We also consider join-principal elements generally under the assumption of the weak union condition and show, for example, that the square of a join-principal element is principal.

Throughout we assume that $\mathcal{L}$ is a local Noether lattice with maximal element $M$.

**Theorem 1.** If $A \in \mathcal{L}$ is join-principal, then, for each $n \geq 1$, $A^n$ has a unique minimal basis in $\mathcal{L}/((0 : A) \wedge A)$.

**Proof.** Use Theorem 1 of [2] to choose a minimal base $E_1, \ldots, E_k$ for $A$ such that $E_i E_j = 0$ whenever $i \neq j$.

Let $F$ be any principal element $\leq A^n \vee ((0 : A) \wedge A)$. Then by Lemma 1 of [3], there exist principal elements $F_1, \ldots, F_{k+1}$ such that

$$F \vee \left( \bigvee_{i \neq j} F_i E_i^n \right) = \bigvee_i F_i E_i^n$$

for $j = 1, \ldots, k + 1$, where $E_{k+1} = 1$ and $F_{k+1} \leq (0 : A) \wedge A$.

Then $FE_j = F_j E_j^{n+1} (1 \leq j \leq k)$, so

$$FA = \bigvee_{1 \leq j \leq k} F_j E_{j}^{n+1} = \bigvee_{1 \leq j \leq k} F_j E_j^n A.$$

Hence

$$F \vee (0 : A) = \left( \bigvee_{1 \leq j \leq k} F_j E_j^n \right) \vee (0 : A).$$

So, since principal elements are join-irreducible,

$$F \vee ((0 : A) \wedge A) = E_j^n \vee ((0 : A) \wedge A),$$

for some $j$. Hence either $F \vee ((0 : A) \wedge A) = E_j^n \vee ((0 : A) \wedge A)$, or $F \leq MA^n \vee ((0 : A) \wedge A)$ and $F$ cannot be used in a minimal base for $A^n$ in
It follows that the only minimal bases for $A^n$ in $\mathcal{E}/((0 : A) \land A)$ are subsets of $E_1^n \lor ((0 : A) \land A), \ldots, E_k^n \lor ((0 : A) \land A)$. On the other hand, if $E_j^n \leq E_1^n \lor \cdots \lor E_j^n \lor \cdots \lor E_k^n \lor ((0 : A) \land A)$, then $E_j^n A = E_j^{n+1} \leq (E_1^n \lor \cdots \lor E_j^n \lor \cdots \lor E_k^n \lor ((0 : A) \land A)) E_j = 0$. Hence $A^n$ has the unique minimal base in $\mathcal{E}/A \land (0 : A)$ consisting of those $E_j^n \lor (A \land (0 : A))$ with $E_j^n A \neq 0$. Q.E.D.

**Corollary 2.** If $\mathcal{E}$ satisfies the weak union condition and if $A \in \mathcal{E}$ is join-principal, then $A = E \lor ((0 : A) \land A)$ for some principal element $E$. In particular, $A^2 = E^2$ is principal.

**Proof.** If the weak union condition is satisfied, only principal elements have unique minimal bases. Since $\mathcal{E}/A \land (0 : A)$ inherits the hypothesis from $\mathcal{E}$, the result follows.

**Corollary 3.** Let $(R, M)$ be a local ring in which $M$ is join-principal. Then, as an $R$-module, $M$ is the direct sum of a cyclic $R$-module and an $R/M$-vector space.

In the general setting, we get the following

**Theorem 4.** If the maximal element $M$ of $\mathcal{E}$ is join-principal, then $\mathcal{E}/(0 : M)$ is distributive.

**Proof.** Let $F$ be any principal element of $\mathcal{E}$, $F \leq 0 : M$. By Theorem 1 and Corollary 1 of [2], there exist principal elements $E_1, \ldots, E_k$ in $\mathcal{E}$ such that $E_1 \lor (0 : M), \ldots, E_k \lor (0 : M)$ are an independent minimal base of $M$ in $\mathcal{E}/(0 : M)$. Choose $n$ so that $F \leq M^n \lor (0 : M)$ and $F \leq M^{n+1} \lor (0 : M)$. Since $M$ is join-principal in $\mathcal{E}$, it follows that $F \lor (0 : M) = E_i^n \lor (0 : M)$ for some $i$. Since the $E_j \lor (0 : M)$ are independent, it follows that $\mathcal{E}/(0 : M)$ is distributive. Q.E.D.

**Corollary 5.** If the maximal element $M$ of $\mathcal{E}$ is join-principal, then every element of $\mathcal{E}$ is join-principal.

**Proof.** Let $B$ and $A$ be any elements of $\mathcal{E}$. Since $M$ is join-principal in $\mathcal{E}/C$ for every $C$, it suffices to show that $BA : A = B \lor (0 : A)$.

Let $E_1, \ldots, E_k$ be independent principal elements such that $E_1 \lor (0 : M), \ldots, E_k \lor (0 : M)$ form the minimal base for $M$ in $\mathcal{E}/0 : M$. Then from the proof of Theorem 4, $AB : A$ and $B \lor (0 : A)$ are both joins of powers of the $E_i \lor (0 : M)$ in $\mathcal{E}/0 : M$.

Set $B \lor (0 : A) = E_1^{s_1} \lor \cdots \lor E_k^{s_k} \lor (0 : M)$, and assume $E_i^n \leq BA : A$. Then $E_i^n A \leq A(B \lor (0 : A)) = AE_1^{s_1} \lor \cdots \lor AE_k^{s_k}$. Since $E_jX$ is a power of $E_j$ for every $X \neq I$ and since the $E_j$ are independent, it follows that $E_i^n A \leq E_j^{s_j} A$. If $n < s_j$, then $A \leq E_i^{s_i-n} A \lor (0 : E_i^n)$. However, in this case, $A \leq 0 : E_i^n$ and $E_i^n \leq 0 : A$. If $n \geq s_j$, then $E_i^n \leq E_i^{s_i} \lor \cdots \lor E_k^{s_k}$. In either case, $E_i^n \leq B \lor (0 : A)$. Hence $BA : A \leq B \lor (0 : A)$. Q.E.D.

**References**

1. E. W. Johnson and J. P. Lediaev, *Structure of Noether lattices into join-principal maximal*

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