A CHARACTERIZATION OF MINIMAL HAUSDORFF SPACES

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ABSTRACT. This paper gives a characterization of minimal Hausdorff spaces.

1. Preliminary definitions and theorems. A net $\mathcal{O} \rightarrow X$ $r$-converges to $x_0 \in X$ if for each open $V$ containing $x_0$, there exists a $d \in \mathcal{O}$ such that $\mathcal{O}(T_d) \subset \text{cl}(V)$ [2]. A net $\mathcal{O} \rightarrow X$ $r$-accumulates to $x_0 \in X$ if for each open $V \subset X$ containing $x_0$ and for every $d \in \mathcal{O}$, $\mathcal{O}(T_d) \cap \text{cl}(V) \neq \emptyset$. Theorem 5 of [1] shows that a Hausdorff space $X$ is minimal Hausdorff if and only if each net in $X$ with a unique $r$-accumulation point is convergent.

A function $f: X \rightarrow Y$ has a strongly-closed graph if for each $(x,y) \in G(f)$ ($G(f)$ denotes the graph of $f$) there exist open sets $U \subset X$ and $V \subset Y$ containing $x$ and $y$, respectively, such that $(U \times \text{cl}(V)) \cap G(f) = \emptyset$ [2]. According to Theorem 7 of [1], each function $f: X \rightarrow Y$ of a topological space $X$ into a minimal Hausdorff space $Y$ with strongly-closed graph is continuous. (Note that Example 3 of [1] shows that the strongly-closed graph condition in Theorem 7 of [1] cannot be relaxed to a closed graph condition.)

2. Main result. Denote by $\mathfrak{S}$ the class of spaces containing the class of Hausdorff completely normal and fully normal spaces [3].

**Theorem.** A Hausdorff space $Y$ is minimal Hausdorff if and only if for every topological space $X$ belonging to $\mathfrak{S}$, each function $f: X \rightarrow Y$ with a strongly-closed graph is continuous.

**Proof.** In view of Theorem 7 of [1], only the sufficiency requires proof. Assume that $Y$ is not minimal Hausdorff. By Theorem 5 of [1] there exists a net $f: \mathcal{O} \rightarrow Y$ with a unique $r$-accumulation point $q \in Y$ such that $f$ does not converge to $q$. Let $\infty \not\in \mathcal{O}$ and define $X = \mathcal{O} \cup \{\infty\}$. Then the power set of $\mathcal{O}$ together with $\{T_d \cup \{\infty\}|d \in \mathcal{O}\}$ is a base for a topology $\sigma$ on $X$ making $(X,\sigma)$ a fully normal, completely normal Hausdorff space [2], [3]. Define $g: X \rightarrow Y$ by $g|\mathcal{O} = f$ and $g(\infty) = q$. Using the fact that $q$ is the unique $r$-accumulation point of the net $f$, it follows that $G(g)$ is strongly-closed. The

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1 The concepts of $r$-convergence and $r$-accumulation point were first introduced by N. V. Veličko under the names of $\beta$-convergence and $\beta$-contact point, respectively, in H-closed topological spaces, Mat. Sb. 70 (112) (1966), 98–112.

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identity function \( \mathcal{D} \rightarrow \mathcal{D} \) defines a net that converges to \( \infty \). However, since \( f \) does not converge to \( q \), there exists an open set \( V \subset Y \) containing \( q \) with the property that \( g(T_d) \cap (Y - V) \neq \emptyset \) for each \( d \in \mathcal{D} \). Consequently, \( g \) is not continuous at \( x = \infty \). This contradiction establishes the proof.

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