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THE SIGNATURE OF THE FIXED SET OF A MAP OF ODD PERIOD

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ABSTRACT. Let T be a diffeomorphism of odd period n on a closed smooth manifold M^{2k}. The Conner-Floyd analysis of fixed point data and the Atiyah-Singer Index Theorem are applied to prove there exist methods of orienting the components F of the fixed set of T, depending only on n, so that $\sum_F \operatorname{sgn} F \equiv \operatorname{sgn} M \bmod 4$ whenever T^* is the identity on $H^k(M; Q)$. Other special results of this type are obtained when assumptions are made restricting the possible eigenvalues in the normal bundle to the fixed set.

Let M^{2n} be a closed oriented differentiable manifold and T an orientation preserving diffeomorphism of M of period k, k an odd integer. Denote by F the fixed point set of T. Our principal result is then

THEOREM 1. *If T^* is the identity on $H^n(M; Q)$, then there is a systematic orientation for F so that*

$$\operatorname{sgn}(M) \equiv \operatorname{sgn}(F) \mod 4.$$

The technique for orienting F depends only on k and not on $[T, M]$. This generalizes considerably Corollary 2.11 of [1], and is the best possible result for manifolds of (positive) dimension $\equiv 0 \bmod 4$. The proof involves the connection between the Atiyah-Singer-Segal G-Signature Theorem [2], [3] and the Conner-Floyd computation of $\mathfrak{M}_*(\mathbf{Z}_k)$, the bordism algebra of local information for $\mathbf{Z}_k$ actions [4], [5].

Denote by $\mathcal{O}_*(\mathbf{Z}_k)$ the bordism ring of orientation preserving actions of $\mathbf{Z}_k$ on closed smooth manifolds and by $\mathfrak{M}_*(\mathbf{Z}_k)$ the bordism ring of actions of $\mathbf{Z}_k$ on compact oriented smooth manifolds with boundary, having no fixed points on the boundary. $\mathfrak{M}_*(\mathbf{Z}_k)$ may be given a bundle theoretic interpretation [4] as follows: We consider bordism classes $[(\xi_1, \ldots, \xi_{(k-1)/2}) \to V]$ of ordered $(k-1)/2$-tuples of complex vector bundles over closed oriented manifolds V. (We allow 0 as a place holder in case ξ_r is the 0-bundle.) If ε: $\{1, 2, \ldots, (k-1)/2\} \to \{\pm 1\}$ is a function and $\mathbf{Z}_k$ acts in ξ_r by multiplication by $\lambda^{\varepsilon(r)\cdot r}$, $\lambda = \exp(2\pi i/k)$, the orientations on the $\{\xi_r\}$ and on V induce an orientation on the disk bundle. The action of $\mathbf{Z}_k$ is fixed point free on the sphere bundle so that the disk bundle gives rise to an element of $\mathfrak{M}_*(\mathbf{Z}_k)$. In fact this correspondence is an isomorphism of bordism theories.

The homomorphism $\operatorname{fix}: \mathcal{O}_*(\mathbf{Z}_k) \to \mathfrak{M}_*(\mathbf{Z}_k)$ is given by $\operatorname{fix}([T, M])$

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$= \sum_F [T_F, (N_F, \partial N_F)]$ where N_F is an invariant normal tube around a component F of the fixed set of T in M. There is a splitting of the normal bundle of F into eigenbundles $(\xi_1, \ldots, \xi_{(k-1)/2}) \to F$ where the eigenvalue for ξ_r is $\lambda^{\varepsilon(r)\cdot r}$ and the orientation on F is induced by the orientations on the $\{\xi_r\}$ and the orientation on N_F given by M. (Clearly different choices of the function ε may produce different orientations of F.) Thus

$$[T_F, (N_F, \partial N_F)] = [(\xi_1, \ldots, \xi_{(k-1)/2}) \to F].$$

The G-signature homomorphism σ: $\mathcal{O}_*(\mathbf{Z}_k) \to \mathbf{Z}(\lambda)$ defined by Atiyah and Singer [3] assigns to the bordism class of an action an algebraic integer that depends only on the fixed point data. This leads to a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_*(\mathbf{Z}_k) \arrow[r, "\sigma"] \arrow[d, "\text{fix}"] & \mathbf{Z}(\lambda) \arrow[d] \\
\mathfrak{M}_*(\mathbf{Z}_k) \arrow[r, "\hat{\sigma}"] & \mathbf{Z}(1/k, \lambda)
\end{tikzcd}$$

In part, the Atiyah-Singer-Segal G-Signature Theorem [2], [3] computes the formula expressing the global invariant $\sigma[T, M]$ in terms of the fixed point information $\hat{\sigma}(\text{fix}\,[T, M])$. The image of $\hat{\sigma}$ is actually contained in the subring of $\mathbf{Z}(\lambda/k)$ given by

$$S = \left\{ \sum_{i=1}^{(k-1)/2} m_i(\lambda^i + \lambda^{-i}) + \sum_{i=1}^{(k-1)/2} n_i(\lambda^i - \lambda^{-i}) \,\middle|\, m_i \equiv m_j \bmod 2, \right.$$

$$\left. n_i \equiv n_j \bmod 2 \text{ and } m_i, n_i \in \mathbf{Z}(1/k) \right\}$$

and is generated [4] by 1 and

$$\left\{ c_r = \frac{\lambda^r + 1}{\lambda^r - 1} \,\middle|\, 1 \leqslant r \leqslant \frac{k-1}{2} \right\}.$$

Now suppose θ: $S \to \mathbf{Z}_4$ is a ring homomorphism such that $\theta(1) = 1$ and $\theta(c_r) = \pm 1$ for each r. There is a corresponding function ε: $\{1, 2, \ldots, (k-1)/2\} \to \{\pm 1\}$ given by $\varepsilon(r) = \theta(c_r)$. As noted previously such a function ε gives rise to a systematic choice of eigenvalues in the normal bundle to a component F of the fixed set and hence a well-defined orientation of F.

To analyze the composition $\mathfrak{M}_*(\mathbf{Z}_k) \xrightarrow{\hat{\sigma}} S \xrightarrow{\theta} \mathbf{Z}_4$ we use the fact that $\mathfrak{M}_*(\mathbf{Z}_k)$ is a polynomial algebra over Ω_*^{SO} on generators

$$y_{r,j} = [(0, \ldots, \xi_r, \ldots, 0) \to CP(j)]$$

where ξ_r is the conjugate Hopf bundle and $\mathbf{Z}_k$ acts in ξ_r by $\lambda^{\varepsilon(r)\cdot r}$ [4]. Applying the commutativity of the diagram (2) to the action on $CP(j+1)$ given by $T_j[z_0, \ldots, z_{j+1}] = [\lambda^{\varepsilon(r)\cdot r} z_0, z_1, \ldots, z_{j+1}]$ we get

$$\hat{\sigma}(y_{r,j}) = \begin{cases} [\hat{\sigma}(y_{r,0})]^{j+1}, & j \text{ even}, \\ 1 - [\hat{\sigma}(y_{r,0})]^{j+1}, & j \text{ odd}. \end{cases}$$

Now $\hat{\sigma}[y_{r,0}] = \varepsilon(r) \cdot c_r$ so

$$\theta(\hat{\sigma}(y_{r,j})) = \begin{cases} 1, & j \text{ even}, \\ 0, & j \text{ odd}, \end{cases}$$

which agrees with sgn $CP(j)$. Therefore the composite $\mathfrak{M}_*(\mathbf{Z}_k) \xrightarrow{\hat{\sigma}} S \xrightarrow{\theta} \mathbf{Z}_4$ is given by

$$\theta\hat{\sigma}[(\xi_1, \ldots, \xi_{(k-1)/2}) \to V] \equiv \operatorname{sgn} V \mod 4.$$

Hence the commutative diagram

$$\begin{tikzcd} \mathcal{O}_*(\mathbf{Z}_k) \arrow[r, "\sigma"] \arrow[d, "\text{fix}"] & S \cap \mathbf{Z}(\lambda) \arrow[d] \arrow[dr, "\theta"] & \\ \mathfrak{M}_*(\mathbf{Z}_k) \arrow[r, "\hat{\sigma}"] & S \arrow[r, "\theta"'] & \mathbf{Z}_4 \end{tikzcd}$$

implies that $\theta(\sigma[T, M]) = \theta(\hat{\sigma}(\text{fix}\,[T, M])) = \sum_F \operatorname{sgn} F \mod 4$. When T^* is the identity on $H^n(M; Q)$, $\sigma[T, M] = \operatorname{sgn} M$. Thus Theorem 1 follows from

PROPOSITION 3. *There are $2^{\varphi(k)/2}$ distinct ring homomorphisms θ: $S \to \mathbf{Z}_4$ such that $\theta(1) = 1$ and $\theta(c_r) = \pm 1$ for each r. (These may not all yield distinct orientations.)*

To prove this let $\beta_i = 2(\lambda^i + \lambda^{-i})$, $\eta_j = 2(\lambda^j - \lambda^{-j})$ and

$$Y = \sum_{j=1}^{(k-1)/2} \lambda^j - \lambda^{-j}.$$

Then S is generated as a $\mathbf{Z}(1/k)$ module by

$$\{\beta_i\}_{i=1}^{(k-1)/2}, \quad \{\eta_j\}_{j=1}^{(k-1)/2}, \quad Y \quad \text{and} \quad -1 = \sum_{i=1}^{(k-1)/2} \lambda^i + \lambda^{-i}.$$

The $\mathbf{Z}(1/k)$-submodule U of S generated by $\{\beta_i, \eta_j | 1 \leqslant i,j \leqslant (k-1)/2\}$ is an ideal. If $x \in S$, denote by $\bar{x}$ the image of x in the quotient $S/2U$. Let $S_{\mathbf{R}} = S \cap \mathbf{R}$ and $A = S_{\mathbf{R}}/2U \cap \mathbf{R}$. As an abelian group $A \simeq \mathbf{Z}_4 \oplus (\mathbf{Z}_2)^{\varphi(k)/2-1}$.

Now A is generated by $\bar{1}$ and $\{\bar{\beta}_i\}_{i=1}^{(k-1)/2}$. Since each $\bar{\beta}_i$ has order two and $\bar{\beta}_i\bar{\beta}_j = 0$, it can be shown that any linear map $\hat{\theta}$: $A \to \mathbf{Z}_4$ such that $\hat{\theta}(\bar{1}) = 1$ is a ring homomorphism. Hence there are $2^{\varphi(k)/2-1}$ ring homomorphisms $\hat{\theta}$: $A \to \mathbf{Z}_4$ with $\hat{\theta}(\bar{1}) = 1$. Noting that $\bar{Y}^2 = 1$, one easily verifies that each such homomorphism admits two extensions $\hat{\theta}$: $S/2U \to \mathbf{Z}_4$ which send $\bar{Y}$ to $+1$ and -1. Thus there are $2^{\varphi(k)/2}$ ring homomorphisms from $S/2U$ to $\mathbf{Z}_4$ sending $\bar{1}$ to 1 and $\bar{Y}$ to ± 1.

It can be checked that for each j, $c_j = \pm Y \mod U$, so all of the above homomorphisms have $\hat{\theta}(\bar{c}_j) = \pm 1$. For each $\hat{\theta}$ define θ to be the composition

$$S \to S/2U \xrightarrow{\hat{\theta}} \mathbf{Z}_4.$$

This completes the proof of Proposition 3.

For certain special classes of actions there is a stronger relation than that given by Theorem 1. Specifically we have the following theorem.

THEOREM 4. *Suppose T is a smooth effective map of odd period k on M^{2n} such that there is only one type of irreducible representation about the fixed set. Then if T^* is the identity on $H^n(M; Q)$,* sgn M *is congruent to* sgn F mod $2^{\varphi(k)}$.

PROOF. Suppose that λ is the eigenvalue corresponding to the one type of irreducible representation. We may as well assume that $\lambda = \exp(2\pi i/k)$. Let $\psi_k(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_0$ be the cyclotomic polynomial for λ, where $m = \varphi(k)$. Define

$$\begin{aligned} f_k(t) &= (t-1)^m \psi_k((t+1)/(t-1)) \\ &= (t+1)^m + a_{m-1}(t+1)^{m-1}(t-1) + \cdots. \end{aligned}$$

Note that $f_k(1) = 2^{\varphi(k)}$ and $f_k((\lambda+1)/(\lambda-1)) = 0$. Also $f_k(0) = \psi_k(-1) = \pm 1$ so that f_k is primitive. Together with the Gauss lemma this implies that the natural homomorphism

$$\mathbf{Z}[t]/\langle f_k(t)\rangle \to Q[t]/\langle f_k(t)\rangle$$

is injective, so we can identify $\mathbf{Z}[t]/\langle f_k(t)\rangle$ with $\mathbf{Z}(x) \subseteq Q(\lambda)$ where $x = (\lambda+1)/(\lambda-1)$. There is also a natural map $\mathbf{Z}[t]/\langle f_k(t)\rangle \to \mathbf{Z}/2^{\varphi(k)}\mathbf{Z}$ given by sending $g(t)$ to $g(1)$.

Let $\overline{\mathfrak{M}}_*(\mathbf{Z}_k)$ denote the subgroup of $\mathfrak{M}_*(\mathbf{Z}_k)$ consisting of those actions with the prescribed representation type about the fixed set. Similarly let $\bar{\mathcal{O}}_*(\mathbf{Z}_k)$ be the subgroup of $\mathcal{O}_*(\mathbf{Z}_k)$ consisting of those actions having this representation about the fixed set and further having $T^* =$ identity on the middle dimensional rational cohomology.

On $\bar{\mathcal{O}}_*(\mathbf{Z}_k)$ we have that $\sigma(T, M) = \operatorname{sgn}(M)$, and $\hat{\sigma}$ restricted to $\overline{\mathfrak{M}}_*(\mathbf{Z}_k)$ takes values in $\mathbf{Z}(x)$ where $x = (\lambda+1)/(\lambda-1)$. Hence we have a commutative diagram

$$\begin{tikzcd}
\bar{\mathcal{O}}_*(\mathbf{Z}_k) \arrow[r, "\mathrm{sgn}"] \arrow[dd] & \mathbf{Z} \arrow[dd] \arrow[dr] & \\
 & & \mathbf{Z}/2^{\varphi(k)}\mathbf{Z} \\
\overline{\mathfrak{M}}_*(\mathbf{Z}_k) \arrow[r] & \mathbf{Z}(x) \arrow[ur] &
\end{tikzcd}$$

in which the composition across the bottom sends an item of fixed point data to the signature of the fixed set mod $2^{\varphi(k)}$. This may be checked on the conjugate Hopf bundle $\eta \to CP(l)$ by recalling [4] that the resulting value is

$$\begin{cases} \left(\dfrac{\lambda+1}{\lambda-1}\right)^{l+1} = x^{l+1} & \text{when } l \text{ is even,} \\ 1 - \left(\dfrac{\lambda+1}{\lambda-1}\right)^{l+1} = 1 - x^{l+1} & \text{when } l \text{ is odd.} \end{cases}$$

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