THE SIGNATURE OF THE FIXED SET OF A MAP OF ODD PERIOD

J. P. ALEXANDER, G. C. HAMRICK AND J. W. VICK

Abstract. Let \( T \) be a diffeomorphism of odd period \( n \) on a closed smooth manifold \( M^{2k} \). The Conner-Floyd analysis of fixed point data and the Atiyah-Singer Index Theorem are applied to prove there exist methods of orienting the components \( F \) of the fixed set of \( T \), depending only on \( n \), so that \( \sum F \text{ sgn } F \equiv \text{ sgn } M \mod 4 \) whenever \( T^* \) is the identity on \( H^k(M; \mathbb{Q}) \). Other special results of this type are obtained when assumptions are made restricting the possible eigenvalues in the normal bundle to the fixed set.

Let \( M^{2n} \) be a closed oriented differentiable manifold and \( T \) an orientation preserving diffeomorphism of \( M \) of period \( k \), \( k \) an odd integer. Denote by \( F \) the fixed point set of \( T \). Our principal result is then

Theorem 1. If \( T^* \) is the identity on \( H^n(M; \mathbb{Q}) \), then there is a systematic orientation for \( F \) so that

\[
\text{sgn}(M) \equiv \text{sgn}(F) \mod 4.
\]

The technique for orienting \( F \) depends only on \( k \) and not on \([T, M]\). This generalizes considerably Corollary 2.11 of [1], and is the best possible result for manifolds of (positive) dimension \( \equiv 0 \mod 4 \). The proof involves the connection between the Atiyah-Singer-Segal G-Signature Theorem [2], [3] and the Conner-Floyd computation of \( \mathfrak{M}_*(\mathbb{Z}_k) \), the bordism algebra of local information for \( \mathbb{Z}_k \) actions [4], [5].

Denote by \( \mathfrak{O}_*(\mathbb{Z}_k) \) the bordism ring of orientation preserving actions of \( \mathbb{Z}_k \) on closed smooth manifolds and by \( \mathfrak{M}_*(\mathbb{Z}_k) \) the bordism ring of actions of \( \mathbb{Z}_k \) on compact oriented smooth manifolds with boundary, having no fixed points on the boundary. \( \mathfrak{M}_*(\mathbb{Z}_k) \) may be given a bundle theoretic interpretation [4] as follows: We consider bordism classes \( [(\xi_1, \ldots, \xi_{(k-1)/2}) \to V] \) of ordered \((k-1)/2\)-tuples of complex vector bundles over closed oriented manifolds \( V \). (We allow 0 as a place holder in case \( \xi_r \) is the 0-bundle.) If \( \epsilon: \{1, 2, \ldots, (k-1)/2\} \to \{\pm 1\} \) is a function and \( \mathbb{Z}_k \) acts in \( \xi_r \) by multiplication by \( \lambda^{\epsilon(r)} \), \( \lambda = \exp(2\pi i/k) \), the orientations on the \( \{\xi_r\} \) and on \( V \) induce an orientation on the disk bundle. The action of \( \mathbb{Z}_k \) is fixed point free on the sphere bundle so that the disk bundle gives rise to an element of \( \mathfrak{M}_*(\mathbb{Z}_k) \). In fact this correspondence is an isomorphism of bordism theories.

The homomorphism \( \text{fix}: \mathfrak{O}_*(\mathbb{Z}_k) \to \mathfrak{M}_*(\mathbb{Z}_k) \) is given by \( \text{fix}([T, M]) \).
\[ T_F, (N_F, \partial N_F) = \sum_F [T_F, (N_F, \partial N_F)] \] where \( N_F \) is an invariant normal tube around a component \( F \) of the fixed set of \( T \) in \( M \). There is a splitting of the normal bundle of \( F \) into eigenbundles \( (\xi_1, \ldots, \xi_{(k-1)/2}) \rightarrow F \) where the eigenvalue for \( \xi_r \) is \( \lambda^{(r)} \) and the orientation on \( F \) is induced by the orientations on the \( \{ \xi_r \} \) and the orientation on \( N_F \) given by \( M \). (Clearly different choices of the function \( \epsilon \) may produce different orientations of \( F \).) Thus

\[ [T_F, (N_F, \partial N_F)] = [(\xi_1, \ldots, \xi_{(k-1)/2}) \rightarrow F]. \]

The \( G \)-signature homomorphism \( \sigma: \Theta_*(Z_k) \rightarrow Z(\lambda) \) defined by Atiyah and Singer [3] assigns to the bordism class of an action an algebraic integer that depends only on the fixed point data. This leads to a commutative diagram

\[
\begin{array}{ccc}
\Theta_*(Z_k) & \xrightarrow{\sigma} & Z(\lambda) \\
\downarrow \text{fix} & & \downarrow \\
\mathcal{M}_*(Z_k) & \xrightarrow{\delta} & Z(1/k, \lambda)
\end{array}
\]

In part, the Atiyah-Singer-Segal G-Signature Theorem [2], [3] computes the formula expressing the global invariant \( \sigma[T, M] \) in terms of the fixed point information \( \sigma(\text{fix} [T, M]) \). The image of \( \delta \) is actually contained in the subring of \( Z(\lambda/k) \) given by

\[
S = \left\{ \left( \sum_{i=1}^{(k-1)/2} m_i (\lambda^i + \lambda^{-i}) + \sum_{i=1}^{(k-1)/2} n_i (\lambda^i - \lambda^{-i}) \right) \mid m_i \equiv m_j \mod 2, \right. \\
\left. n_i \equiv n_j \mod 2 \text{ and } m_i, n_i \in Z(1/k) \right\}
\]

and is generated [4] by 1 and

\[
\left\{ c_r = \frac{\lambda^r + 1}{\lambda^r - 1} \mid 1 \leq r \leq \frac{k - 1}{2} \right\}.
\]

Now suppose \( \theta: S \rightarrow Z_4 \) is a ring homomorphism such that \( \theta(1) = 1 \) and \( \theta(c_r) = \pm 1 \) for each \( r \). There is a corresponding function \( \epsilon: \{1, 2, \ldots, (k - 1)/2\} \rightarrow \{\pm 1\} \) given by \( \epsilon(r) = \theta(c_r) \). As noted previously such a function \( \epsilon \) gives rise to a systematic choice of eigenvalues in the normal bundle to a component \( F \) of the fixed set and hence a well-defined orientation of \( F \).

To analyze the composition \( \mathcal{M}_*(Z_k) \xrightarrow{\delta} S \xrightarrow{\theta} Z_4 \) we use the fact that \( \mathcal{M}_*(Z_k) \) is a polynomial algebra over \( \Omega^{SO}_* \) on generators

\[
y_{r,j} = [(0, \ldots, 0, \xi_r, \ldots, 0) \rightarrow CP(j)]
\]

where \( \xi_r \) is the conjugate Hopf bundle and \( Z_k \) acts in \( \xi_r \) by \( \lambda^{(r)} \) [4]. Applying the commutativity of the diagram (2) to the action on \( CP(j + 1) \) given by

\[
T_j[z_0, \ldots, z_{j+1}] = [\lambda^{(r)} z_0, z_1, \ldots, z_{j+1}] \]

we get
\[
\hat{\sigma}(y_{r,j}) = \begin{cases} 
[\hat{\sigma}(y_{r,0})]^{j+1}, & \text{j even,} \\
1 - [\hat{\sigma}(y_{r,0})]^{j+1}, & \text{j odd.}
\end{cases}
\]

Now \(\hat{\sigma}[y_{r,0}] = e(r) \cdot c, \) so

\[
\theta(\hat{\sigma}(y_{r,j})) = \begin{cases} 
1, & \text{j even,} \\
0, & \text{j odd,}
\end{cases}
\]

which agrees with \(\operatorname{sgn} CP(j).\) Therefore the composite \(\mathfrak{M}_*(Z_k) \xrightarrow{\hat{\sigma}} S \xrightarrow{\theta} Z_4\) is given by

\[
\theta(\hat{\sigma}[(\xi_1, \ldots, \xi_{(k-1)/2}) \to V]) \equiv \operatorname{sgn} V \mod 4.
\]

Hence the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{M}_*(Z_k) & \xrightarrow{\sigma} & S \cap Z(\lambda) \\
\downarrow \text{fix} & & \downarrow \theta \\
\mathfrak{M}_*(Z_k) & \xrightarrow{\hat{\sigma}} & S \\
\end{array}
\]

implies that \(\theta(\sigma[T, M]) = \theta(\hat{\sigma}(\text{fix } [T, M])) = \sum F \operatorname{sgn} F \mod 4.\) When \(T^*\) is the identity on \(H^n(M; Q), \sigma[T, M] = \operatorname{sgn} M.\) Thus Theorem 1 follows from

**Proposition 3.** There are \(2^{\varphi(k)/2}\) distinct ring homomorphisms \(\theta: S \to Z_4\) such that \(\theta(1) = 1\) and \(\theta(c_r) = \pm 1\) for each \(r.\) (These may not all yield distinct orientations.)

To prove this let \(\beta_i = 2(\lambda^i + \lambda^{-i}), \eta_j = 2(\lambda^j - \lambda^{-j})\) and

\[
Y = \sum_{j=1}^{(k-1)/2} \lambda^j - \lambda^{-j}.
\]

Then \(S\) is generated as a \(Z(1/k)\) module by

\[
\{ \beta_i \}_{i=1}^{(k-1)/2}, \quad \{ \eta_j \}_{j=1}^{(k-1)/2}, \quad Y\text{ and } -1 = \sum_{i=1}^{(k-1)/2} \lambda^i + \lambda^{-i}.
\]

The \(Z(1/k)\)-submodule \(U\) of \(S\) generated by \(\{ \beta_i, \eta_j \}_{1 \leq i, j \leq (k-1)/2}\) is an ideal. If \(x \in S, \) denote by \(\bar{x}\) the image of \(x\) in the quotient \(S/2U.\) Let \(S_R = S \cap R\) and \(A = S_R/2U \cap R.\) As an abelian group \(A \simeq Z_4 \oplus (Z_2)^{\varphi(k)/2-1} - 1.\)

Now \(A\) is generated by \(\bar{1}\) and \(\{ \bar{\beta}_i \}_{i=1}^{(k-1)/2}.\) Since each \(\bar{\beta}_i\) has order two and \(\bar{\beta}_i \bar{\beta}_j = 0,\) it can be shown that any linear map \(\hat{\theta}: A \to Z_4\) such that \(\hat{\theta}(\bar{1}) = 1\) is a ring homomorphism. Hence there are \(2^{\varphi(k)/2-1}\) ring homomorphisms \(\hat{\theta}: A \to Z_4\) with \(\hat{\theta}(\bar{1}) = 1.\) Noting that \(\bar{Y}^2 = 1,\) one easily verifies that each such homomorphism admits two extensions \(\hat{\theta}: S/2U \to Z_4\) which send \(\bar{Y}\) to \(+1\) and \(-1.\) Thus there are \(2^{\varphi(k)/2}\) ring homomorphisms from \(S/2U\) to \(Z_4\) sending \(\bar{1}\) to \(\bar{1}\) and \(\bar{Y}\) to \(\pm 1.\)

It can be checked that for each \(j, c_j = \pm Y \mod U,\) so all of the above homomorphisms have \(\hat{\theta}(c_j) = \pm 1.\) For each \(\hat{\theta}\) define \(\theta\) to be the composition
This completes the proof of Proposition 3.

For certain special classes of actions there is a stronger relation than that given by Theorem 1. Specifically we have the following theorem.

**Theorem 4.** Suppose $T$ is a smooth effective map of odd period $k$ on $M^{2n}$ such that there is only one type of irreducible representation about the fixed set. Then if $T^*$ is the identity on $H^n(M; \mathbb{Q})$, $\text{sgn } M$ is congruent to $\text{sgn } F$ mod $2^{\varphi(k)}$.

**Proof.** Suppose that $\lambda$ is the eigenvalue corresponding to the one type of irreducible representation. We may as well assume that $\lambda = \exp(2\pi i/k)$. Let $\psi_k(t) = t^m + a_{m-1}t^{m-1} + \cdots + a_0$ be the cyclotomic polynomial for $\lambda$, where $m = \varphi(k)$. Define

$$f_k(t) = (t - 1)^m \psi_k((t + 1)/(t - 1))$$

$$= (t + 1)^m + a_{m-1}(t + 1)^{m-1}(t - 1) + \cdots.$$

Note that $f_k(1) = 2^{\varphi(k)}$ and $f_k((\lambda + 1)/(\lambda - 1)) = 0$. Also $f_k(0) = \psi_k(-1) = \pm 1$ so that $f_k$ is primitive. Together with the Gauss lemma this implies that the natural homomorphism

$$\mathbb{Z}[t]/\langle f_k(t) \rangle \rightarrow \mathbb{Q}[t]/\langle f_k(t) \rangle$$

is injective, so we can identify $\mathbb{Z}[t]/\langle f_k(t) \rangle$ with $\mathbb{Z}(x) \subseteq \mathbb{Q}(\lambda)$ where $x = (\lambda + 1)/(\lambda - 1)$. There is also a natural map $\mathbb{Z}[t]/\langle f_k(t) \rangle \rightarrow \mathbb{Z}/2^{\varphi(k)}\mathbb{Z}$ given by sending $g(t)$ to $g(1)$.

Let $\mathfrak{M}_*(\mathbb{Z}_k)$ denote the subgroup of $\mathfrak{M}_*(\mathbb{Z}_k)$ consisting of those actions with the prescribed representation type about the fixed set. Similarly let $\mathfrak{G}_*(\mathbb{Z}_k)$ be the subgroup of $\mathfrak{G}_*(\mathbb{Z}_k)$ consisting of those actions having this representation about the fixed set and further having $T^*$ = identity on the middle dimensional rational cohomology.

On $\mathfrak{G}_*(\mathbb{Z}_k)$ we have that $\sigma(T, M) = \text{sgn } (M)$, and $\hat{\sigma}$ restricted to $\mathfrak{M}_*(\mathbb{Z}_k)$ takes values in $\mathbb{Z}(x)$ where $x = (\lambda + 1)/(\lambda - 1)$. Hence we have a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{G}_*(\mathbb{Z}_k) & \xrightarrow{\text{sgn}} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathfrak{M}_*(\mathbb{Z}_k) & \rightarrow & \mathbb{Z}(x)
\end{array}
\]

in which the composition across the bottom sends an item of fixed point data to the signature of the fixed set mod $2^{\varphi(k)}$. This may be checked on the conjugate Hopf bundle $\eta \rightarrow CP(l)$ by recalling [4] that the resulting value is

\[
\begin{align*}
\left(\frac{\lambda + 1}{\lambda - 1}\right)^{l+1} & = x^{l+1} & \text{when } l \text{ is even}, \\
1 - \left(\frac{\lambda + 1}{\lambda - 1}\right)^{l+1} & = 1 - x^{l+1} & \text{when } l \text{ is odd}.
\end{align*}
\]
REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TEXAS 78712