A NOTE ON IDENTIFICATIONS OF METRIC SPACES

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ABSTRACT. A space $X$ is said to be $oMK$ provided that $X$ has a countable closed cover $\mathcal{C}$ of metrizable subspaces such that if $K$ is a compact subset of $X$, there is a $C \in \mathcal{C}$ for which $K \subseteq C$. A Hausdorff space is $oMK$ and Fréchet if and only if it is representable as a closed image of a metric space obtained by identifying a discrete collection of closed sets with hemicompact boundaries to points.

A familiar example of a nonmetrizable space is $R/N$, that is, the space obtained by identifying the set of natural numbers $\mathbb{N}$ in the set of real numbers $\mathbb{R}$ to a point and giving the resulting set the quotient topology. In [5], the concept of a $oMK$ space proved useful in characterizing certain countably infinite spaces. This note relates identification spaces such as $R/N$ with the concept of a $oMK$ space.

All spaces in this paper are understood to be Hausdorff topological spaces and all mappings are continuous onto functions. A space $X$ is $oMK$ provided that $X$ has a countable closed cover $\mathcal{C}$ of metrizable subspaces such that if $K$ is a compact subset of $X$, there is a $C \in \mathcal{C}$ for which $K \subseteq C$. We may assume that $\mathcal{C}$ consists of sets $C_1 \subseteq C_2 \subseteq \cdots$, and we will henceforth do so. A space $X$ is Fréchet [2] provided that every accumulation point of a set $A$ in $X$ is the limit of some sequence in $A$. It is clear that $oMK$ and Fréchet are each hereditary properties.

Theorem 1. If a space $X$ is $oMK$ and Fréchet, then it is an image of a metric space $M$ under a closed mapping $f$, and there is a discrete collection $\mathcal{F}$ of closed subsets of $M$ such that $f(F)$ is a point for each $F \in \mathcal{F}$, $\text{Bdy} F$ is hemicompact for each $F \in \mathcal{F}$, and $f$ is one-to-one upon restriction to $M - \bigcup \mathcal{F}$.

The proof follows from a number of propositions.

The concept of a $oMK$ space arose in analogy to the concept of hemicompactness introduced by Arens [1]. A space $X$ is hemicompact provided that there is a countable cover $\mathcal{C}$ of compact subspaces such that if $K$ is a compact subset of $X$, there is a $C \in \mathcal{C}$ for which $K \subseteq C$. A metrizable space is hemicompact if and only if it is separable and locally compact.

Proposition 2. (a) If a space $X$ is $oMK$ and Fréchet, then it is a closed image of a metric space having cardinality that of $X$.

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(b) A space $X$ is hemicompact, Fréchet, and has every compact subspace metrizable if and only if it is a closed image of a locally compact separable metric space. (The metric space may be chosen to have cardinality that of $X$.)

**Proof.** We need only prove case (a), since case (b) is similar. (The “if” of case (b) is well known. In part, see [3].) Thus, assume that $X$ is Fréchet and $X = \bigcup \{C_n \mid n \in N\}$ as given in the definition of $\sigma MK$.

Let $C_0 = \emptyset$. Let $M_n = \text{cl}(C_n - C_{n-1})$ for each $n$. Let $M$ be the discrete union of the $M_n$ and let $f$ be the natural mapping of $M$ onto $X$. Clearly $M$ is metrizable and $f$ is continuous. We need to show that $f$ is closed. Let $x_0$ be a point of $X$ for which there is a sequence $\{x_i\}$ of distinct points of $X - \{x_0\}$ converging to $x_0$. It suffices to show that if points $p_i \in f^{-1}(x_i)$ are chosen for each $i$, then the sequence $\{p_i\}$ has a convergent subsequence in the space $M$. So, let $p_i \in f^{-1}(x_i)$ for each $i$.

We will need the fact that there exists an integer $n_0$ for which $\{x_0, x_1, x_2, \ldots\}$ is contained in $\bigcup \{M_n \mid n = 1, 2, \ldots, n_0\}$, and $x_i \notin M_n$ for $i \in N$ and $n > n_0$. Suppose not. Then there exists a subsequence $\{x_i\}$ of $\{x_j\}$ and a subsequence $\{n_j\}$ of $N$ such that $x_{n_j} \in \text{cl}(C_{n_j+1} - C_{n_j})$ for each $j$, with $n_j \neq n_k$ for distinct $j, k$. Then for each $j$, there is a sequence $\{x_{n_j}^k \mid k \in N\}$ $\subset C_{n_j+1} - C_{n_j}$ such that $x_{n_j}^k \to x_{n_j}$. By the Fréchet assumption, since $x_0$ is an accumulation point of the set $\{x_{n_j}^k \mid k \in N\}$, there is a sequence $\{q_j\}$ contained in $\{x_{n_j}^k \mid k \in N\}$ such that $q_j \to x_0$ and $q_j \neq x_0$ for all $j$. Let $F$ be the set $\{q_j \mid j \in N\}$. Then for each $n$, $F \cap C_n$ is finite and so closed. Since $X$ is $\sigma MK$ and Fréchet, $F$ must be closed. This is impossible, so our supposition is false.

By the fact that we have just shown, there is a subsequence $\{x_{i_j}\}$ and there is an integer $n_1 \leq n_0$ for which $\{x_0, x_{i_1}, x_{i_2}, \ldots\}$ is contained in $M_{n_1}$, and for each $n > n_1$, $x_{i_j} \in M_n$ for at most finitely many $j \in N$. Thus there is an $n_2 \leq n_1$ for which a subsequence of $\{p_1, p_2, \ldots\}$ is contained in $M_{n_2}$. This subsequence of $\{p_1, p_2, \ldots\}$ converges in $M_{n_2}$, and so also in $M$.

**Lemma 3.** Suppose $S = \{0, i, (i, j, k) \mid i, j, k \in N\}$ has a topology with the following properties. Each point $(i, j, k)$ is itself an open set. Each set $S_i = \{i, (i, j, k) \mid j, k \in N\}$ is an open set and is homeomorphic to the “sequential fan” (that is, a set $G$ is a neighborhood of $i$ in $S_i$ if $i \in G$, and for each $j$, $(i, j, k) \in G$ for all but finitely many $k$). The sequence $i$ converges to 0. Then $S$ cannot be both $\sigma MK$ and Fréchet.

**Proof.** Suppose on the contrary that $S$ is Fréchet and $S = \bigcup \{C_i \mid i \in N\}$ as given by the definition of $\sigma MK$. We may assume (without loss of generality) that $\{0, 1, 2, \ldots\} \subset C_1$. Notice that for each $i$, $S_i$ is not contained in $C_i$, since $S_i$ is not metrizable. In fact, for each $i$, $S_i - C_i$ must contain a sequence $S'_i = \{(i, j, k) \mid n \in N\}$ for some $j \in N$ and some subsequence $\{k_n\}$ of $\{k \mid k \in N\}$. Let $S' = \bigcup \{S'_i \mid i \in N\}$. Then 0 is an accumulation point of $S'$. Since $S$ is assumed to be Fréchet, there is a sequence $T$ in $S'$ which converges to 0. But then, there is an integer $i_0$ for which $T \subset C_{i_0}$. Since $T \subset S'$ and $T$ converges to 0, there is a point $x_0$ common to $T$ and $\bigcup \{S'_i \mid i \geq i_0\}$. Let $i_1 \geq i_0$ be such that $x_0 \in T \cap S'_{i_1}$. Then $x_0 \in T \subset C_{i_0} \subset C_{i_1}$. Thus, $x_0 \in S'_{i_1} \subset C_{i_1}$. This is a contradiction.

**Proposition 4.** If a space $X$ is $\sigma MK$ and Fréchet, and $D$ is the set of those
points of $X$ at which $X$ is not first-countable, then no point of $X$ is an accumulation point of $D$.

**Proof.** Otherwise, there exists a sequence $\{x_n\}$ of distinct points converging to a point $x_0 \in X$ such that each $x_n$ (for $n = 1, 2, \ldots$) is a point of non-first-countability and $x_n \neq x_0$. There exists a sequence of disjoint open sets $G_n$ such that $x_n \in G_n$ for $n = 1, 2, \ldots$. Since each $G_n$ is Fréchet, but not countably bisequential at $x_n$ (since a closed image of a metric space which is countably bisequential is metrizable), by [6], there exists a copy of the sequential fan in $G_n$ "at $x_n"$. Let $S_n$ denote this copy. Thus, for each $n = 1, 2, \ldots$, there is an $S_n$ "at $x_n"," and these $S_n$ are disjoint. Let $S = \bigcup \{S_n\}_{n=1,2,\ldots} \cup \{x_0\}$. By Lemma 3, $S$ is either not $\sigma MK$ or not Fréchet. We have a contradiction.

**Proposition 5.** If a space $X = M/F$, where $X$ is $\sigma MK$, $M$ is metrizable, and $F$ is a closed subset of $M$, then Bdy $F$ is hemicompact.

**Proof.** If Bdy $F$ is empty, it is trivially hemicompact. If Bdy $F$ is nonempty, $M/F = (M - \text{Int } F)/\text{Bdy } F$, so we may assume (without loss of generality) that the interior of $F$ is empty. Let $X = \bigcup \{C_n\}_{n \in \mathbb{N}}$ as given by the definition of $\sigma MK$. Let $f$ be the natural mapping of $M$ into $X$. Since there exists an $n \in \mathbb{N}$ for which the point $f(F) \in C_n$, we may assume that in fact, $f(F) \subseteq C_1$. Since $C_n$ is metrizable, $f_n = f|f^{-1}(C_n)$ is a closed mapping of $f^{-1}(C_n)$ onto $C_n$ with Bdy $f_n^{-1}(x)$ being compact for each point $x$ of $C_n$ ([4] or [7]). Let $K_n = \text{Bdy } f_n^{-1}(f(F))$ for each $n$, where the boundary is taken relative to $f^{-1}(C_n)$. Then each $K_n$ is compact. Also, $F = \bigcup \{K_n\}_{n \in \mathbb{N}}$. Because, if $p \in F$, there is a sequence $\{p_n\}$ in $M - F$ which converges to $p$. But there is an integer $n_0$ for which $\{f(p), f(p_1), f(p_2), \ldots\} \subseteq C_{n_0}$. Thus, $p \in \text{cl } (f^{-1}(C_{n_0}) - F) \cap F = K_{n_0}$.

In order to show that $\{K_n\}_{n \in \mathbb{N}}$ is a sequence as in the definition of hemicompact, let $K$ be a compact subset of $F$. Suppose that for all $n$, there is a point $x_n \in K - K_n \subseteq K - \text{cl } (f^{-1}(C_n) - F)$. Since $K$ is sequentially compact, there is a point $x$ in $K$ which is the limit of some subsequence of $\{x_n\}$. For simplicity of notation, assume that the subsequence is $\{x_n\}$ itself. Let $\{G_n\}_{n \in \mathbb{N}}$ be an open base at $x$ in $M$ such that $G_n \supseteq G_{n+1}$ for each $n$. Since $x_n$ is in (the boundary of) $F$, there exists a sequence $\{x^n_m\}_{m \in \mathbb{N}}$ in $G_n - F$ which converges to $x_n$. Let $L = \{x, x_n, x^n_m | m, n \in \mathbb{N}\}$. Then $L$ is a compact subset of $M$. Since $f(L)$ is compact in $X$, there is an integer $n_0$ for which $f(L) \subseteq C_{n_0}$. Then the sequence $\{x^n_{n_0}\}_{m \in \mathbb{N}}$ is contained in $f^{-1}(C_{n_0}) - F$. This means that $x_{n_0} \in \text{cl } (f^{-1}(C_{n_0}) - F)$. We have a contradiction, and thereby we have shown that (the boundary of) $F$ is hemicompact.

**Proof of Theorem 1.** Let $X$ be $\sigma MK$ and Fréchet. By Proposition 2, $X$ is a closed image of a metric space $M'$ under a mapping $h$. Let $D$ be the set of those points of $X$ at which $X$ is not first-countable. By Proposition 4, $D$ is a discrete closed subspace of $X$, and ([4] or [7]) $X - D$ is metrizable. Let $M = h^{-1}(D) \cup (X - D)$, define $g: M' \to M$ by $g|h^{-1}(D) = \text{identity}$, $g|X - D = f|X - D$, define $h: M \to X$ by $f|h^{-1}(D) = h|h^{-1}(D)$, $f|(X - D) = \text{identity}$. Giving $M$ the quotient topology as an image of $M'$, $g$...
and \( f \) are closed continuous mappings and \( M \) is metrizable. Let \( \mathcal{F} = \{ f^{-1}(x) \mid x \in D \} \). Then \( \mathcal{F} \) is a discrete collection in \( M \) and by Proposition 5, each \( F \in \mathcal{F} \) has a hemicompact boundary. The proof is complete.

We now prove the converse of Theorem 1.

**Theorem 6.** If a space \( X \) is an image of a metric space \( M \) under a closed mapping \( f \), and there is a discrete collection \( \mathcal{F} \) of closed subsets of \( M \), such that \( f(F) \) is a point for each \( F \in \mathcal{F} \), \( \text{Bdy } F \) is hemicompact for each \( F \in \mathcal{F} \), and \( f \) is one-to-one upon restriction to \( M - \bigcup \mathcal{F} \), then \( X \) is \( oMK \) and Fréchet.

**Proof.** It is well known that a closed image of a metric space is Fréchet. We prove that \( X \) is \( oMK \) under the stated hypotheses. We may assume (without loss of generality) that \( \text{Int } F = \emptyset \) for each \( F \in \mathcal{F} \). Let each \( F = \bigcup \{ C_F^i \mid i \in N \} \) as given by the definition of hemicompact, and we may assume that each \( C_F^i \subset C_F^{i+1} \). Since \( \mathcal{F} \) is a discrete collection in \( M \), there exists a discrete collection \( \{ G_F^i \mid F \in \mathcal{F} \} \) of open sets such that \( F \subset G_F^i \) for every \( F \in \mathcal{F} \). Also, let \( G_F^n = G_F^i \cap S_{1/n}(F - C_F^i) \), where \( S_{1/n}(A) \) denotes the \( 1/n \) open sphere around set \( A \). Let \( D_F^n = C_F^i - G_F^n \). Finally, let \( M'_n = M - \bigcup \{ G_F^i \mid F \in \mathcal{F} \} \). Then \( f(M'_n) \) is closed for each \( n \). Since \( f \) is a closed mapping and \( M'_n \) is a closed subset of \( M \), each \( f(M'_n) \) is a closed mapping. Also, \( f^{-1}(x) \) is compact for each point \( x \) of \( X \), since each \( D_F^n \) is compact. By [4] or [7], each \( M_n \) is then metrizable. It is clear that \( X = \bigcup \{ M_n \mid n \in N \} \).

Now let \( K \) be a compact subset of \( X \). And suppose that \( K \subset M_n \) for all \( n \). Then for each \( n \), there is a point \( x_n \in K - M_n \). But \( K \) is sequentially compact (being compact and Fréchet), and so there is a subsequence \( \{ x_{n_i} \} \) of distinct points, converging to a point \( x \) of \( K \), with \( x_{n_i} \neq x \) for all \( i \). Fix \( m \in N \). Since the sequence \( \{ x_{n_i} \} \) meets \( M_m \) in at most finitely many points, there exists an \( i_m \in N \) such that \( x_{n_i} \in f(\bigcup \{ C_F^i \mid F \in \mathcal{F} \}) \) for all \( i > i_m \). For each \( i > i_m \), let \( y_{n_i} \in f^{-1}(x_{n_i}) \cap \bigcup \{ C_F^i \mid F \in \mathcal{F} \} \). Since the set \( \{ y_{n_i} \mid i > i_m \} \) is not closed in \( M \), let \( y \) be an accumulation point of this set. So there exists a subsequence of \( \{ y_{n_i} \} \) which converges to \( y \). This means that \( f(y) = x \). Also, \( y \in \text{cl } \bigcup \{ G_F^i \mid F \in \mathcal{F} \} \subset \bigcup \{ G_{F_{i+1}}^i \mid F \in \mathcal{F} \} \) for all \( m \). We then have that \( y \in F_0 \) for some \( F_0 \in \mathcal{F} \). Since \( y \in G_{F_0}^0 \subset S_{1/m}(F_0 - C_{F_0}) \) for every \( m \), let \( p_{m} \) be a point of \( F_0 - C_{F_0} \) for which \( d(y, p_m) < 1/m \). Then \( p_m \to y \) in \( F_0 \). So there is an integer \( n_0 \) for which \( \{ y, p_1, p_2, \ldots \} \subset C_{F_0}^{n_0} \). Thus \( p_{n_0} \in C_{F_0}^{n_0} \). This is a contradiction.

**Examples 7.** To illustrate the results of this paper we consider two examples of countable regular Fréchet spaces. Let \( Q \) denote the usual space of rational numbers. Let \( Q' \) be \( Q^2 \) in the plane together with the entire \( x \)-axis of real numbers, with the usual topology from the plane. Let \( X_1 \) be the quotient space obtained by considering \( Q' \) and identifying the \( x \)-axis to a point. Let \( X_2 \) be the quotient space obtained by considering \( Q^2 \) and identifying the set \( Q \) in the \( x \)-axis to a point. Then \( X_1 \) and \( X_2 \) are the desired spaces. By Theorem 6, \( X_1 \) is \( oMK \). On the other hand, \( X_2 \) is not \( oMK \). To see this, suppose \( X_2 = \bigcup \{ C_n \mid n \in N \} \) as given by the definition of \( oMK \) and suppose that \( C_n \subset C_{n+1} \) for all \( n \). Since \( X_2 \) is not metrizable, for each \( n \) there exists a point \( x_n \in \left( \left( [0, l/n]^2 \cap Q^2 \right) / Q \right) - C_n \). Then \( x_n \to (0, 0) \). So there exists an integer \( n_0 \) for which \( \{ x_n \mid n \in N \} \subset C_{n_0} \), contradiction.
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