SHORTER NOTES

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A NEW PROOF OF GIESY'S THEOREM ON NON-B-CONVEXITY

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Abstract. A new proof of Giesy's theorem on non-B-convexity is given.

The following theorem was proved by D. P. Giesy in [2, Lemma 6, §1] and has been often used since (see, for example [1]):

A necessary and sufficient condition for non-D-convexity of a normed linear space $X$ is that given $k > 2, \varepsilon > 0$, there exist $x_1, x_2, \ldots, x_k \in S$ such that for each choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_k$,

$$(1 - \varepsilon) \sum_{i=1}^{k} |\alpha_i| < \left\| \sum_{i=1}^{k} \alpha_i x_i \right\|.$$

It is known that $D$-convexity is equivalent to $B$-convexity [2]. In this note a simple proof for the necessity part is given. The sufficiency is trivial, of course.

The notation is the same as that of Giesy [2].

Proof of the necessity. Let $k > 2$ and $0 < \varepsilon < 1$. Let $\delta = k^{-1} \varepsilon$.

$X$ is not $D$-convex $\Rightarrow X$ is not $D, x, \delta$-convex

$\Rightarrow \exists x_1, x_2, \ldots, x_k \in S$ for any $\lambda_1, \lambda_2, \ldots, \lambda_k \in D$,

$$k^{-1} \left\| \sum_{i=1}^{k} \lambda_i x_i \right\| > 1 - \delta.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be scalars such that $\sum_{i=1}^{k} |\alpha_i| = 1$. Consider

$$\left\| \sum_{i=1}^{k} \alpha_i x_i \right\| \geq \left\| \sum_{i=1}^{k} sg(\alpha_i) x_i \right\| - \left\| \sum_{i=1}^{k} (sg(\alpha_i) - \alpha_i) x_i \right\|$$

$$> k(1 - \delta) - \sum_{i=1}^{k} |sg(\alpha_i) - \alpha_i| = k - \varepsilon - \sum_{i=1}^{k} (1 - |\alpha_i|)$$

$$= k - \varepsilon - (k - 1) = 1 - \varepsilon.$$
Hence \((1 - \varepsilon) \sum_{i=1}^{k} |\alpha_i| \leq \|\sum_{i=1}^{k} \alpha_i x_i\|\) for any choice of scalars \(\alpha_1, \alpha_2, \ldots, \alpha_k\).

References


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