A NOTE ON GENERALIZED RESOLVENTS FOR
ORDINARY DIFFERENTIAL OPERATORS

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ABSTRACT. We give an explicit construction for the kernel of an arbitrary
generalized resolvent for an ordinary symmetric differential operator. In
particular, this avoids the use of approximation of selfadjoint operators on
compact intervals. We also discuss integrability of functions which are
adjoint to certain fundamental solutions.

Coddington [2] showed that the kernel of an arbitrary generalized resolvent
for a symmetric ordinary differential operator is the sum of two kernels: One
is obtained by approximating selfadjoint operators on compact intervals and
the other is obtained by considering the range of the resolvent (see [2, (2.4)];
also [1, p. 179]). In this paper we will give an explicit construction for the
kernel of an arbitrary generalized resolvent as above. In particular, this
avoids the use of approximation of selfadjoint operators. We also discuss
integrability of functions which are adjoint to certain fundamental solutions.

Unexplained notation and terminology will be as in [2]. Let $L$ denote the
formally selfadjoint differential operator $P_0(x)D^n + P_1(x)D^{n-1} + \cdots +
P_n(x)$ where $D = d/dx$, the $P_k$ are complex-valued functions of class $C^{n-k}$
on an open interval $(a, b)$ ($-\infty < a < b < \infty$) and $P_0(x) \neq 0$ on $(a, b)$. This
operator $L$ generates a minimal symmetric operator $T_0$ and a maximal
operator $T$ in the Hilbert space $\mathcal{H} = L^2(a, b)$. Let $R(l)$ denote a generalized
resolvent for $T_0$. Then $R$ is given by

$$R(l) = (T_{\lambda(l)} - l)^{-1}(\Re l > 0); \quad R(\bar{l}) = (R(l))^*.$$  

**Theorem 1 (Coddington [2]).** The generalized resolvent $R(l)$ is an integral
operator of Carleman type having a kernel $K = K(x, y, l)$ with the property that
$\partial_j x^{j-k} K / \partial x^{j-k} y^{k-1}$ ($j, k = 1, \ldots, n$) are continuous in $(x, y, l)$ and analytic
in $l$ on any region for which $\Re l \neq 0$, except for $x = y$ when $j$ or $k$ is $n$.

Our purpose here is to compute $K$ explicitly. Let $c$ be an arbitrary, but
fixed point in $(a, b)$. Let $\alpha(l)$ denote the maximum number of linearly
independent $L^2(a, c)$-solutions of $Lf = lf$. Let $\beta(l)$ denote the maximum
number of linearly independent $L^2(c, b)$-solutions of $Lf = lf$. Finally let $\omega(l)$
denote the maximum number of linearly independent $L^2(a, b)$-solutions of
$Lf = lf$. It is well known that $\alpha(l)$, $\beta(l)$, $\omega(l)$ only depend on whether $\Re l > 0$

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or $g/l < 0$. When $g/l > 0$, we put $\omega(l) = \omega^+, \omega(l) = \omega^-$. It is shown in [3] and [5] that $\alpha(l) + \beta(l) = \omega(l) + n$ for $g/l \neq 0$. Using this equality and linear independence, we can choose fundamental solutions $\xi_i(x, l), \ldots, \xi_n(x, l)$ of $Lf = Lf$ ($g/l \neq 0$) such that $\{\xi_i(x, l): 1 < i < \omega(l)\} \subset \mathcal{C}^2(a, b); \{\xi_i(x, l): \omega(l) < i < \alpha(l)\} \subset \mathcal{C}^2(a, c); \{\xi_i(x, l): \alpha(l) < i < n\} \subset \mathcal{C}^2(c, b)$ (cf. [8, pp. 90–91]). Let

$$\xi_i(y, l) = (-1)^{n+i}W_i(\xi_1, \ldots, \xi_n)(y)/P_0(y)W(\xi_1, \ldots, \xi_n)(y)$$

for $1 < i < n$. Here $W_i(\xi_1, \ldots, \xi_n)(y)$ denotes the determinant obtained from the Wronskian $W(\xi_1, \ldots, \xi_n)(y)$ of $\xi_1(y, l), \ldots, \xi_n(y, l)$ by removing its $n$th row and $i$th column. Let $\lambda$ be an arbitrary, but fixed complex number with $g\lambda > 0$. For any $f$ in $\mathcal{C}^2(a, b)$ with compact support on $(a, b)$, set $u = R(\lambda)f$. Then $(T_A^\lambda - \lambda)u = (L - \lambda)u = f$. Thus, using the variation of constants formula and the definitions of $\alpha(\lambda), \beta(\lambda)$ and $\omega(\lambda)$, we have

$$R(\lambda)f = \sum_{i=1}^{\omega^+} \xi_i(x, \lambda)a_i + \int_a^b \Omega_0(x, y, \lambda)f(y) \, dy$$

for some complex constants $a_i$. Here for $g/l \neq 0$,

$$\Omega_0(x, y, l) = \begin{cases} -\sum_{l=1}^{\omega(l)} \xi_i(x, l)\xi_{*}(y, l), & x < y, \\ \sum_{l=1}^{\omega(l)} \xi_i(x, l)\xi_{*}(y, l), & x > y. \end{cases}$$

Since $R(\lambda)f \in \mathcal{D}(\lambda)$, we have $\langle R(\lambda)f \rangle(\lambda) = 0$ for $j = 1, 2, \ldots, \omega^+$ (see [2, p. 381]). In view of (1), this can be written

$$\sum_{i=1}^{\omega^+} a_i\langle \xi_i(\cdot, \lambda)v_i^*(\cdot, \lambda)\rangle = -\int_a^b \langle \Omega_0(\cdot, y, \lambda)v_i^*(\cdot, \lambda)\rangle f(y) \, dy$$

for $j = 1, 2, \ldots, \omega^+$. Let $\Delta(\lambda)$ denote the $\omega^+ \times \omega^+$ matrix with $(j, i)$ entry $\langle \xi_i(\cdot, \lambda)v_j^*(\cdot, \lambda)\rangle$. Clearly $\Delta(\lambda)$ is nonsingular. Let $\xi^+(x, \lambda)$ denote the $1 \times \omega^+$ row vector $(\xi_i(x, \lambda))$ and let $V_*(y, \lambda)$ denote the $\omega^+ \times 1$ column vector $(v_j^*(y, \lambda))$. Then substituting the solutions $(a_i)$ to (2) into (1), we get

$$R(\lambda)f = \int_a^b \Omega(x, y, \lambda)f(y) \, dy$$

where

$$\Omega(x, y, \lambda) = \Omega_0(x, y, \lambda) - \xi^+(x, \lambda)\Delta^{-1}(\lambda)\langle \Omega_0(\cdot, y, \lambda)V_*(\cdot, \lambda)\rangle.$$

Similarly, we have

$$(R(\lambda))^*f = \int_a^b \Omega(x, y, \lambda)f(y) \, dy$$

where

$$\Omega(x, y, \lambda) = \Omega_0(x, y, \lambda) - \xi^-(x, \lambda)\Delta^{-1}(\lambda)\langle \Omega_0(\cdot, y, \lambda)V_*(\cdot, \lambda)\rangle.$$
we see that $\Omega(x, y, \lambda) = \overline{\Omega}(y, x, \overline{\lambda})$. From this symmetry it can be shown easily that $\Omega(x, y, \lambda)$ belongs to $L^2(a, b)$ as a function of $y$. Thus, since $R(\lambda)$ is a bounded operator, (3) holds for every $f \in L^2(a, b)$. Hence together with Theorem 1 we have $\int_a^b (K(x, y, \lambda) - \Omega(x, y, \lambda))f(y) \, dy = 0$ for any $f \in L^2(a, b)$. This yields

**Theorem 2.** $K(x, y, l) = \Omega(x, y, l)$ ($\notin l \neq 0$).

In view of $\Omega$ and using the definitions of $\alpha(l)$, $\beta(l)$ and $\omega(l)$, $\Omega$ can be written

$$\Omega(x, y, l) = \Omega_0(x, y, l) + \sum_{l=1}^{\omega(l)} \xi_l(x, l)\theta_l(y, l)$$

for $\notin l \neq 0$. Since $\Omega$ is symmetric, for $y < x$, we have

$$\sum_{l=1}^{\omega(l)} \xi_l(x, l)\left[\xi_{l-1}(y, l) + \theta_l(y, l)\right] + \sum_{l=1}^{\omega(l)} \xi_l(x, l)\xi_{l+1}(y, l) = 0$$

Since the $\xi_l(x, l)$ are linearly independent, we have

**Theorem 3.** For $\notin l \neq 0$,

$$\{\xi_l(y, l): \alpha(l) < i < n\} \cup \{\xi_l(y, l) + \theta_l(y, l): 1 < i < \omega(l)\}$$

is linearly independent, and

$$\sum_{l=1}^{\omega(l)} \xi_l(y, l)\theta_l(x, l) = 0$$

Here l.s. denotes "the linear span of".

When the left endpoint $a$ is regular, $\theta_l(y, l) \in L^2(a, b)$ for $1 < i < \omega(l)$. Thus in view of (4) we have

**Corollary.** If the left endpoint of $(a, b)$ is a regular point, for each $l$ with $\notin l \neq 0$, the map $f \mapsto \int_a^b \Omega_0(x, y, l)f(y) \, ds$ is a bounded operator on $L^2(a, b)$.

**References**


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