THE SPACES \text{lip} \alpha AND CERTAIN OTHER SPACES HAVE DUALS WITH CES\'ARO BASES

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Abstract. Banach sequence spaces whose duals are Banach sequence spaces with Toeplitz bases are characterized. For example, the duals of the \text{lip} \alpha spaces, for $0 < \alpha < 1$, are shown to have Ces\'aro bases. Also reflexive spaces with a Toeplitz basis are characterized and an equivalent form of the well-known theorem of F. and M. Riesz on the absolute continuity of measures is given.

1. Introduction. Let $0 < \alpha < 1$, let $f_h(x) = f(x + h)$, and let $\| \cdot \|$ denote the sup norm of $L_\infty(0, 2\pi)$. Lip \alpha is the space of all $2\pi$-periodic complex valued functions on $\mathbb{R}$ for which $\|f_h - f\| = O(|h|^\alpha)$ as $h \to 0$, and lip \alpha is the space of all $f \in \text{Lip} \alpha$ for which $\|f_h - f\| = o(|h|^\alpha)$ as $h \to 0$. Lip \alpha and lip \alpha are Banach spaces under the norm

$$\|f\|_\alpha = \max\left\{\|f\|, \sup_{h \neq 0} \frac{\|f_h - f\|}{|h|^\alpha}\right\}.$$ 

In [2], K. de Leeuw showed that the second dual $(\text{lip} \alpha)^{''}$ of lip \alpha is Lip \alpha. His proof also shows that the dual of lip \alpha is separable. We prove general theorems on Banach sequence spaces which show, in particular, that (lip \alpha)$'$ even has a Ces\'aro basis. In general, we characterize Banach sequence spaces whose duals are Banach sequence spaces with Toeplitz bases ($\S$3). We apply these theorems to lip \alpha in 4.1 and 4.2, to reflexive Banach spaces with Toeplitz bases in 4.3, and to the well-known F. and M. Riesz Theorem in 4.5.

This paper is a result of several discussions on the lip \alpha spaces with Professor Günther Goes to whom I am also thankful for many suggestions.

2. Definitions. Let $\omega$ be the space of all complex valued sequences $x = (x_k)_{k=1}^\infty$ and let $\phi$ be the subspace of all sequences with a finite number of nonzero coordinates. Let $T = (t_{nk})$ be an infinite matrix with rows in $\phi$ and with columns converging to 1. For each $x \in \omega$, the $n$th $T$-section of $x$ is $t^nx = (t_{nk}x_k)$. A BK-space is a subspace of $\omega$ which is a Banach space with continuous coordinates [10], [9, §§11.3, 12.4]. We suppose all BK-spaces considered contain $\phi$.

Let $E$ be a BK-space. $E_{AD}$ is the closure of $\phi$ in $E$. $E_{TB}$ is the space of all sequences $x$ in $\omega$ (not necessarily in $E$) for which $\sup_n \|t^nx\|_E < \infty$ and $E_{TK}$ is the space of all sequences $x$ in $E$ for which $\lim_n \|t^nx - x\|_E = 0$. $E_{FTK}$

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\[ \{ x \in \omega : \lim_{n \to \infty} f(t^n x) \text{ exists for all } f \in E' \}. \]

Clearly \( E_{TK} \subseteq E_{AD} \) and \( E_{TK} \subseteq E_{FK} \subseteq E_{TB} \). \( E \) is a TB-space (resp., a TK-space, an AD-space) if \( E \subseteq E_{TB} \) (resp., \( E = E_{TK} \), \( E = E_{AD} \)).

The condition TK says that \( \{ t^n \}_{n=1}^\infty \) is an approximate identity for \( E \). This implies that the set \( \{ e_k \}_{k=1}^\infty \), where \( e_k \) is the sequence with 1 in the \( k \)th position and zero elsewhere, is a Toeplitz basis for \( E \) [5], [6, p. 45]. In particular, let \( T = \sigma = (\sigma_{nk}) \) be the matrix defined by \( \sigma_{nk} = (1 - (k - 1)/n) \) if \( k \leq n \) and \( \sigma_{nk} = 0 \) if \( k > n \). Then \( \{ e_k \} \) is a Cesàro basis (of order one) for \( E \) if and only if \( E \) is a \( \sigma \)-space.

Let \( E \) be a BK-space. Each \( f \in E' \) defines a sequence \( y_f = (f(e_k))_{k=1}^\infty \). Thus \( E' \) is associated with the sequence space \( E^\phi = \{ y \in \omega : \exists \zeta \in E' \exists y = y_f \} \). If the correspondence \( f \to y_f \) is one-to-one, then we can identify \( E' \) with the sequence space \( E^\phi \) and we write \( E' \equiv E^\phi \). Since \( E_{AD} \) is a BK-space and \( (E_{AD})' = (E_{AD})^\phi = E^\phi \), \( E \) is a BK-space under the dual norm of \( E_{AD} \) [1, Proposition 1]. If \( E' \equiv E^\phi \), then we identify \( E'' \) with the dual of the BK-space \( E^\phi \). Thus \( E'' \equiv (E^\phi)^\phi \) denotes \( (E^\phi)' \equiv (E^\phi)^\phi \). We define \( E^{\tau \tau} \) (respectively, \( E^{\beta \tau} \)) as the space of all \( y \) in \( \omega \) for which \( \sum_{k=1}^\infty t_{nk} x_k y_k \) is bounded (resp., converges as \( n \to \infty \)) for each \( x \) in \( E \).

3. Main results.

3.1. Proposition. Let \( E \) be a BK-space. Then \( E' \equiv E^\phi \) if and only if \( E \) is an AD-space.

Proof. We have \( E' \equiv E^\phi \) if and only if every \( f \) in \( E' \) is determined by its values on \( e_k^k, k = 1, 2, 3, \ldots \). Since \( \phi \) is the span of \( \{ e_k \}_{k=1}^\infty \), this is equivalent to the property AD. \( \square \)

3.2. Corollary. Suppose \( E \) is a BK-space and an AD-space. Then \( E'' \equiv (E^\phi)^\phi \) if and only if \( E^\phi \) is an AD-space.

3.3. Theorem. Let \( E \) be a BK-space. Consider the following statements.

(a) \( E \) is a TB-space.
(b) \( E_{TB} = (E^{\tau \tau})^{\tau \tau} \).
(c) \( E^\phi \) is a TB-space.
(d) \( E_{AD} \) is a TK-space.
(e) \( E_{TB} = (E^\phi)^\phi \).

We have the following implications:

\[ (a) \iff (b) \iff (c) \iff (d) \iff (e). \]

If \( E \) is an AD-space, all statements are equivalent.

Proof. (a)\( \iff (b) \) follows from Theorem 4 of [1]. A TB-space which is an AD-space is a TK-space [11, Satz 4]. Hence (a) \( \Rightarrow (d) \). By [1, Theorem 1], \( E_{TB} = (E^\phi)^{\tau \tau} \) and by [1, Theorem 4], \( E^\phi \) is a TB-space if and only if \( (E^\phi)^{\tau \tau} = (E^\phi)^\phi \). Thus (c) \( \iff (e) \). (c) \( \iff (d) \) follows from \( E^\phi = (E_{AD})^\phi \) and [1, Proposition 2]. If \( E = E_{AD} \), then evidently (d) \( \Rightarrow (a) \). \( \square \)

3.4. Remark. If \( E \) is not an AD-space the implication (c) \( \Rightarrow (a) \) in Theorem 3.3 is false. For example, let \( E \) be the space spanned by the constant sequences
and the space \( \ell = \{ x \in \omega : \sum |x_k| < \infty \} \). Then \( E \) is a BK-space under the norm given by \( \|z\|_E = |c(z)| + \sum |z_k - c(z)| \) where \( c(z) = \lim_k z_k \). Let \( t_{nk} = 1 \) if \( k < n \) and \( t_{nk} = 0 \) if \( k > n \). Then \( E \) is not a TB-space since \( E_{AD} = E_{TK} = E_{TB} = \ell \). But \( E^\phi = (E_{AD})^\phi = f^\phi = \{ x \in \omega : \sup_k |x_k| < \infty \} \) is a TB-space.

3.5. Corollary. Let \( E \) be a BK-space. Then \( E \) is a TK-space if and only if \( E' \cong E^\phi \) and \( E^\phi \) is a TB-space.

3.6. Theorem. Suppose \( E \) is a BK-space and an AD-space. Then \( E'' \cong (E^\phi)^\phi \) = \( E_{TB} \) if and only if \( E^\phi \) is a TK-space.

Proof. \( E^\phi \) is a TK-space if and only if it is an AD-space and a TB-space [11, Satz 4]. By 3.2, \( E^\phi \) is an AD-space if and only if \( E'' \cong (E^\phi)^\phi \). By 3.3, \( E^\phi \) is a TB-space if and only if \( (E^\phi)^\phi = E_{TB} \).

3. Applications. \( \text{Lip} \alpha \) and \( \text{lip} \alpha \) can be identified with the BK-spaces \( \text{Lip}^\alpha \) and \( \text{lit}^\alpha \) consisting of the sequences \( \hat{f} \) of Fourier coefficients of \( f \) in \( \text{Lip} \alpha \) and \( \text{lip} \alpha \), respectively, when we define the norm by \( \|\hat{f}\|_\alpha = \|f\|_\alpha \). These are spaces of sequences \( x = (x_k)_{k=\omega}^{+\infty} \) defined on the integers rather than the natural numbers. However, the results of §3 hold for these sequence spaces with minor changes in some definitions. In [7], it is shown that \( (\text{Lip}^\alpha)_{AD} = \text{lip}^\alpha \). Thus, by 3.1, \( (\text{lip} \alpha)' \) can be identified with the BK-space \( (\text{lip}^\alpha)^\phi \).

4.1. Theorem. \( (\text{lip} \alpha)' \) is a Banach space with a Cesàro basis.

Proof. We show that \( (\text{lip}^\alpha)^\phi \) is a \( \sigma K \)-space. Let \( f \in \text{Lip} \alpha \). We define \( \|f\|_{\alpha, \infty} = \|f\| \), where \( \|\cdot\| \) denotes the sup norm on \( L^\infty(0, 2\pi) \). Since \( \sup_n \|\alpha^n f\|_{L^\infty} = \|\hat{f}\|_{L^\infty} \), [12, p. 137], we have

\[
\sup_n \|\alpha^n \hat{f}\|_{\alpha, \infty} = \sup_n \max \left\{ \|\alpha^n \hat{f}_h - \alpha^n \hat{f}\|_{L^\infty}, \sup_{h \neq 0} \frac{\|\alpha^n (f_h - \hat{f})\|_{L^\infty}}{|h|^\alpha} \right\} = \max \left\{ \sup_n \|\alpha^n \hat{f}\|_{L^\infty}, \sup_{h \neq 0} \frac{\|\alpha^n (f_h - \hat{f})\|_{L^\infty}}{|h|^\alpha} \right\} = \|\hat{f}\|_{\alpha, \infty}.
\]

Hence \( (\text{lip}^\alpha)_{\sigma K} = \text{Lip}^\alpha \). De Leeuw [2] has shown that \( (\text{lip} \alpha)'' \) is \( \text{Lip} \alpha \). By 3.6, \( (\text{lip}^\alpha)^\phi \) is a \( \sigma K \)-space.

4.2. Remark. It follows from 3.5 that \( \text{lip} \alpha \) also has a Cesàro basis. For TK-spaces, the \( \beta_T \) and \( \gamma_T \) duals are equal [1, Theorem 5]. Therefore, it follows that \( \text{Lip}^\alpha = ((\text{lip}^\alpha)^\phi)^\beta = (\text{lip}^\alpha)_{F, K} \).

The following characterization of reflexive Banach spaces with Toeplitz bases implies Theorems 7 and 8 of [3] and Theorem 3′ of [8].

4.3. Theorem. Let \( E \) be a BK-space and a TK-space. \( E \) is reflexive if and only if \( E = (E_{\beta_T})_{\beta_T} \) and \( E^\phi \) is a TK-space.

Proof. If \( E \) is a reflexive TK-space, then \( E'' \cong (E^\phi)^\phi = E \). By 3.2, \( E^\phi \) is an AD-space. By 3.5, \( E^\phi \) is also a TB-space. Hence \( E^\phi \) is a TK-space [11, Satz 4]. By [1, Theorem 5], \( E = (E_{\gamma_T})_{\gamma_T} = (E_{\beta_T})_{\beta_T} \). Conversely, if \( E \) and \( E^\phi \) are
TK-spaces, then by 3.2, 3.3, and [1, Theorem 5],

\[ E'' \cong (E^\phi)^{\beta_T} = E_{TB} = (E^{\gamma_T})^{\gamma_T} = (E^{\beta_T})^{\beta_T}. \]

If also \( E = (E^{\beta_T})^{\beta_T} \), then \( E'' \cong (E^\phi)^{\beta_T} = E \). □

4.4. Remark. It is clear from the proof that the condition \( E = (E^{\beta_T})^{\beta_T} \) in Theorem 4.3 can be replaced by \( E = (E^{\gamma_T})^{\gamma_T} \) or by \( E = E_{TB} \). Also, by 3.5, the condition \( E^\phi \) is a TK-space can be replaced by \( E^\phi \) is an AD-space.

Let \( L, L^\infty, C, \) and \( M \) be the Banach spaces of the \( 2\pi \)-periodic, integrable, essentially bounded measurable, continuous functions and bounded Borel measures, respectively. If \( E \) is one of these spaces, we define

\[ \hat{E}_c = \left\{ x \in \omega : \sum_{k=1}^{\infty} x_k \cos kt \text{ is the Fourier series of an } f \in E \right\} \]

and

\[ \hat{E}_s = \left\{ x \in \omega : \sum_{k=1}^{\infty} x_k \sin kt \text{ is the Fourier series of an } f \in E \right\}. \]

The spaces \( \hat{E}_c \) and \( \hat{E}_s \) are BK-spaces under the norm \( \|x\| = \|f\|_E \) if \( f \) is the generating function (measure) of \( x \). The theorem of F. and M. Riesz [12, p. 285] can be written \( \hat{M}_c \cap \hat{M}_s = \hat{L}_c \cap \hat{L}_s \). Goes [4, Satz 1.22] has given characterizations of this equation in terms of the condition \( E \in \mathcal{K} \). Using 3.6, we easily obtain the following.

4.5. Theorem. The second dual of the BK-space \( \hat{C}_c + \hat{C}_s \) can be identified with \( \hat{L}_c^\infty + \hat{L}_s^\infty \) and this is an equivalent form of the F. and M. Riesz Theorem.

Proof. Let \( E = \hat{C}_c + \hat{C}_s \). By Fejér’s Theorem and [4, Satz 1.12], \( E \) is a \( \sigma \)-K-space. Thus

\[ E' \cong E^\phi = E^{\beta_0} = (\hat{C}_c)^{\beta_0} \cap (\hat{C}_s)^{\beta_0} = \hat{M}_c \cap \hat{M}_s \]

and \( E_{\sigma B} = \hat{L}_c^\infty + \hat{L}_s^\infty \) [4, Beispiele 1.19(b)]. By 3.6, \( E'' \cong (E^\phi)^{\beta_T} = E_{\sigma B} = \hat{L}_c^\infty + \hat{L}_s^\infty \) if and only if \( E^\phi = \hat{M}_c \cap \hat{M}_s \) is a \( \sigma \)-K-space. But since \( \hat{M}_c \cap \hat{M}_s \) is a \( \sigma \)-K-space if and only if

\[ \hat{M}_c \cap \hat{M}_s = (\hat{M}_c \cap \hat{M}_s)_{\sigma K} = (\hat{M}_c)_{\sigma K} \cap (\hat{M}_s)_{\sigma K} = \hat{L}_c \cap \hat{L}_s, \]

our statement is indeed equivalent to the F. and M. Riesz Theorem. □

References


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