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HOMOTOPY LIMITS AND THE HOMOTOPY TYPE OF FUNCTOR CATEGORIES

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ABSTRACT. Let $\mathbf{Y}$: $I \to$ Simplicial Sets be a functor. We give a sufficient condition for the map $\underrightarrow{\text{holim}}\ \mathbf{Y} \to \underrightarrow{\lim}\ \mathbf{Y}$ to be a weak equivalence. Then we apply this to determine the Artin-Mazur homotopy type of the functor category Funct(I, Sets).

1. **Homotopy direct limits.** Let I be a small category, and let $\mathbf{Y}$: $I \to \mathbb{S}$ be a functor ($\mathbb{S}$ is the category of simplicial sets). In [3, XII 2.1 and 3.7], Bousfield and Kan define $\underrightarrow{\text{holim}}\ \mathbf{Y}$, the homotopy direct limit of $\mathbf{Y}$ (Bousfield and Kan work with the pointed category $\mathbb{S}_*$, but they remark [3, XII 3.7] that everything remains true in the unpointed case). They also construct a natural map $\underrightarrow{\text{holim}}\ \mathbf{Y} \to \underrightarrow{\lim}\ \mathbf{Y}$. Proposition 1 below gives a sufficient condition for this map to be a weak equivalence.

First, we need some notation. The "underlying space" or "nerve" of a small category I is denoted $N(I)$ (this differs from the notation of [3, XI 2.1]). If i is an object of I, then $I \backslash i$ is the category of all maps $i \to j$ in I (see [3, XI 2.7]), and we have the formula:

$$N(I\backslash i)_q = \coprod_{u \in N(I)_q} \operatorname{Hom}_I(i, i_q), \quad u = (i_0 \leftarrow \cdots \leftarrow i_q). \tag{1}$$

And given a functor $\mathbf{Y}$: $I \to \mathbb{S}$, we get the functors $\mathbf{Y}_n$: $I \to$ Sets (for $n \geqslant 0$), which are defined as follows: if i is an object of I, then $\mathbf{Y}_n(i)$ is just $\mathbf{Y}(i)_n$, the n-simplices of $\mathbf{Y}(i)$.

PROPOSITION 1. *Let $\mathbf{Y}$: $I \to \mathbb{S}$ be a functor, and assume that each $\mathbf{Y}_n$ is a coproduct of representable functors. Then the natural map*:

$$\underrightarrow{\text{holim}}\ \mathbf{Y} \to \underrightarrow{\lim}\ \mathbf{Y} \tag{2}$$

is a weak equivalence.

PROOF. By assumption, each $\mathbf{Y}_n$ can be written as:

$$\mathbf{Y}_n = \coprod_{\alpha \in A_n} \operatorname{Hom}_I(i_\alpha, \ ) \tag{3}$$

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where A_n is some set. Since the direct limit of any representable functor is a singleton set, we see that $(\varinjlim \mathbf{Y})_n = \varinjlim \mathbf{Y}_n \simeq A_n$.

Next let us look at $\operatorname{holim}_{\rightarrow} \mathbf{Y}$. We define the bisimplicial set $Y_{..}$ as follows:

(4) $$Y_{\cdot q} = \coprod_{u \in N(I)_q} \mathbf{Y}(i_q), \quad u = (i_0 \leftarrow \cdots \leftarrow i_q)$$

where the boundaries and degeneracies are as in [3, XII 5.1] (and $Y_{..}$ is called $\coprod_* \mathbf{Y}$). Then, by [3, XII 5.2], we have:

(5) $$\operatorname{holim}_{\rightarrow} \mathbf{Y} = \operatorname{diag}(Y_{..}).$$

Using (4), (3) and (1), we obtain:

$$\begin{aligned} Y_{pq} &= \coprod_{u \in N(I)_q} \mathbf{Y}_p(i_q) = \coprod_{u \in N(I)_q} \coprod_{\alpha \in A_p} \operatorname{Hom}_I(i_\alpha, i_q) \\ &= \coprod_{\alpha \in A_p} \coprod_{u \in N(I)_q} \operatorname{Hom}_I(i_\alpha, i_q) = \coprod_{\alpha \in A_p} N(I \backslash i_\alpha)_q \end{aligned}$$

and one sees easily that $Y_{p\cdot} = \coprod_{\alpha \in A_p} N(I \backslash i_\alpha)$.

Let $W_{..}$ be the bisimplicial set where $W_{pq} = (\varinjlim \mathbf{Y})_p \simeq A_p$ (so that $W_{p\cdot} \simeq \coprod_{\alpha \in A_p} *$, $*$ being the trivial simplicial set). The obvious maps $N(I \backslash i_\alpha) \to *$ give us maps:

(6) $$Y_{p\cdot} = \coprod_{\alpha \in A_p} N(I \backslash i_\alpha) \to \coprod_{\alpha \in A_p} * = W_{p\cdot}$$

so that we have a bisimplicial map $Y_{..} \to W_{..}$. And we see that the map (2) above is just the map $\operatorname{diag}(Y_{..}) \to \operatorname{diag}(W_{..})$. So we need to prove that it is a weak equivalence.

By [3, XII 3.4], we have a commutative diagram:

(7) $$\begin{array}{ccc} \operatorname{holim}_{\rightarrow} Y_{..} & \to & \operatorname{holim}_{\rightarrow} W_{..} \\ \downarrow & & \downarrow \\ \operatorname{diag}(Y_{..}) & \to & \operatorname{diag}(W_{..}) \end{array}$$

where the vertical arrows are weak equivalences. But each $N(I \backslash i_\alpha)$ is contractible (see [3, XI 2.4]), so that the maps (6) are weak equivalences. Thus, [3, XII 4.2] says that the top arrow of (7) is a weak equivalence, which forces the bottom arrow to be one too. Q.E.D.

2. **The homotopy type of a functor category.** As before, let I be a small category. We want to determine the Artin-Mazur homotopy type of the functor category Funct(I, Sets) (which we will denote $I^\wedge$; see [1, I 1.2]).

What does it mean for $I^\wedge$ to have a homotopy type? Looking at [2, §9], we see that we need the following two things for $I^\wedge$: a Grothendieck topology and a connected component functor.

The first of these is easy to describe: a map $f\colon F \to G$ in $I^\wedge$ is a covering iff for all i in I, $f(i)$ is onto (this is the canonical topology of the topos $I^\wedge$ [1, IV 2.6]).

If one analyzes the connected component functor of [2, §9], one sees that it is nothing but a left adjoint to the constant sheaf functor (see [1, IV 7.6 and

8.7] for full details). But in the case of $I^{\wedge}$, a constant sheaf is just a constant functor on I, and the left adjoint is well known to be direct limit, $\varinjlim$.

Let $HR(I^{\wedge})$ be the homotopy category of hypercoverings of $I^{\wedge}$ (see [2, 8.4 and 8.13]). Then, following [2, §9], we have:

DEFINITION. The homotopy type of $I^{\wedge}$, denoted $\{I^{\wedge}\}_{ht}$, is the pro-object $\{\varinjlim U_{\cdot}\}_{U_{\cdot} \in HR(I^{\wedge})}$ in pro-$\mathcal{H}$ (where $\mathcal{H}$ is the homotopy category).

PROPOSITION 2. *There is a canonical isomorphism*:

$$\{I^{\wedge}\}_{ht} \simeq N(I) \tag{8}$$

in pro-$\mathcal{H}$.

PROOF. First, note that a simplicial object $U_{\cdot}$ of $I^{\wedge}$ can be regarded as a functor $U_{\cdot}: I \to \mathcal{S}$ and then $\varinjlim U_{\cdot}$ has the same meaning it had in §1.

We will use $e_{\cdot}$ to denote the trivial simplicial object of $I^{\wedge}$ (each e_n is the functor which takes all of I to the same one element set). Note that for any simplicial object $U_{\cdot}$ of $I^{\wedge}$, there is a unique map $U_{\cdot} \to e_{\cdot}$.

Let $U_{\cdot}$ be a hypercovering of $I^{\wedge}$. Then, by [2, 8.5(a)], each $U_{\cdot}(i)$ (for i in I) is a contractible Kan complex, so that for each i in I, the map $U_{\cdot}(i) \to e_{\cdot}(i)$ is a weak equivalence. Then, by [3, XII 4.2], the map:

$$\underset{\to}{\operatorname{holim}}\, U_{\cdot} \to \underset{\to}{\operatorname{holim}}\, e_{\cdot} \tag{9}$$

is a weak equivalence. Formulas (4) and (5) show that $\underset{\to}{\operatorname{holim}}\, e_{\cdot}$ is just $N(I)$, so that we get a weak equivalence $\underset{\to}{\operatorname{holim}}\, U_{\cdot} \simeq N(I)$.

Next, assume that each U_n is a coproduct of representable functors. Then, by Proposition 1, the map:

$$\underset{\to}{\operatorname{holim}}\, U_{\cdot} \to \varinjlim U_{\cdot} \tag{10}$$

is a weak equivalence. So for such $U_{\cdot}$'s, there is a canonical weak equivalence $\varinjlim U_{\cdot} \simeq N(I)$. If we can show that these $U_{\cdot}$'s are cofinal in $HR(I^{\wedge})$, then the proposition will be proved.

For any F in $I^{\wedge}$ and i in I, there is a natural transformation of functors $F(i) \times \operatorname{Hom}_I(i, \ ) \to F$ (where the pair $(x,f) \in F(i) \times \operatorname{Hom}_I(i,j)$ gets mapped to $F(f)(x) \in F(j)$). Putting these together, we get a map:

$$\coprod_{i \in obI} F(i) \times \operatorname{Hom}_I(i, \ ) \to F \tag{11}$$

which is a covering in $I^{\wedge}$. This shows that anything in $I^{\wedge}$ can be covered by a coproduct of representable functors. Then, using the techniques of [2, §8], one easily proves that hypercoverings with the above property are cofinal in $HR(I^{\wedge})$. Q.E.D.

REMARK. One can also do a pointed version of this. Let i_0 be a fixed object of I. We get a point of $I^{\wedge}$ (in the sense of [2, §8]), p: $I^{\wedge} \to$ Sets, defined by the formula $p(F) = F(i_0)$. Then we get the pointed homotopy type of $I^{\wedge}$, still denoted $\{I^{\wedge}\}_{ht}$, in pro-$\mathcal{H}_*$, which turns out to be just $N(I)$ with i_0 as the distinguished vertex.

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