SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

ON A LIPSCHITZ ESTIMATE FOR CONFORMAL MAPS IN THE PLANE

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By combining a weak form of Koebe’s \(|\cdot|\)-Theorem with elementary facts about subharmonic functions, one obtains a brief proof of the following statement.

**Theorem.** Suppose \(D\) and \(D^*\) are bounded domains in \(\mathbb{C}\) with \(C^2\) boundary. Then any conformal map \(f: D \to D^*\) is Lipschitz, i.e. there is \(K < \infty\), such that \(|f(z) - f(\xi)| \leq K|z - \xi|\) for \(z, \xi \in D\).

By a \(C^2\) boundary I mean that \(\partial D\) is a not necessarily connected, compact, 1-dimensional \(C^2\) submanifold of \(\mathbb{C}\), or, equivalently, that \(\partial D\) consists of finitely many disjoint closed Jordan curves whose parametrizations in terms of arclength are twice continuously differentiable.

Stronger results, with more complicated proofs, have been known for a long time; for a more recent proof, one may consult Warschawski [4], where one can also find references to the older literature. I thank Professor H. Grunsky for pointing out this reference to me.

**Proof.** Fix \(z \in D\) and restrict \(f\) to the disc with center \(z\) and radius \(d(z, \partial D)\). By Koebe’s Theorem [1, Chapter IV, Satz 59], the image of this disc contains a disc with center \(f(z)\) and radius \(\rho \cdot |f'(z)| \cdot d(z, \partial D)\), where \(\rho > 0\) is independent of \(z\). This implies

\[
(*) \quad |f'(z)| \leq \frac{1}{\rho} \frac{d(f(z), \partial D^*)}{d(z, \partial D)^{-1}} \quad \text{for } z \in D.
\]

Choose a \(C^2\) function \(\phi\) in a neighborhood of \(\overline{D}\), such that \(\phi < 0\) on \(D\), \(\phi = 0\) on \(\partial D\) and \(\text{grad } \phi \neq 0\) on \(\partial D\). Locally, the existence of \(\phi\) is an immediate consequence of the definition of a \(C^2\) submanifold, and the global existence follows by a partition of unity argument. Replacing \(\phi\) by \(\exp(A\phi) - 1\) with sufficiently large \(A > 0\), one can also assume that \(\partial^2\phi / \partial z \partial \overline{z} > 0\) near \(\partial D\). Moreover, there is \(k < \infty\) such that

\[
(**) \quad |\phi(z)| \leq k \cdot d(z, \partial D) \quad \text{for } z \in D.
\]

As suggested by Henkin [2], one now uses the following lemma.

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Lemma (cf. [3]). Suppose \( D \subset \mathbb{C} \) has \( C^2 \) boundary. Let \( u \) be continuous on \( \overline{D} \), \( u < 0 \) on \( D \), \( u = 0 \) on \( \partial D \), and \( u \) is subharmonic on \( U \cap D \) for some neighborhood \( U \) of \( \partial D \). Then there is \( c > 0 \), such that

\[
|u(w)| \geq c \cdot d(w, \partial D) \quad \text{for } w \in D.
\]

Applying this Lemma to \( \phi \circ f^{-1} \) and \( D^* \), together with (**) , one obtains

\[
d(f(z), \partial D^*) \leq k/c \cdot d(z, \partial D).
\]

This estimate and (*) show that \( |f'| \) is bounded, and hence \( f \) is Lipschitz. Q.E.D.

Proof of the Lemma. Choose \( r > 0 \), such that for each \( \xi \in \partial D \) the interior \( \Delta_{\xi}(r) \) of the circle of radius \( r \) through \( \xi \) with center on the inner normal to \( \partial D \) at \( \xi \) lies in \( U \cap D \). Let \( M = \sup \{u(z): z \in D, d(z, \partial D) > r\} \); \( M \) is < 0. For \( z \in D \) with \( d(z, \partial D) < r \) choose \( \xi \in \partial D \) with \( |z - \xi| = d(z, \partial D) \). Let \( P_r \) be the Poisson kernel for the disc \( \Delta_{\xi}(r) \), and let \( d\sigma \) be arclength on \( \partial \Delta_{\xi}(r) \). Then

\[
u(z) \leq \int_{\partial \Delta_{\xi}} u(\lambda) P_r(z, \lambda) d\sigma(\lambda) \leq M \int_{\partial \Delta_{\xi} \cap \{z: d(z, \partial D) > r\}} P_r(z, \lambda) d\sigma(\lambda),
\]

and the Lemma now follows directly from properties of \( P_r \).

References


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