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PRODUCTS OF COUNTABLY COMPACT SPACES

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ABSTRACT. Extensions of sufficient conditions for the product of two countably compact spaces to be countably compact, plus a relevant example.

In 1953 Novák [4] published an example to show that countable compactness is not preserved under products. Novák's example consists of taking two countably compact subspaces A_1 and A_2 of the Stone-Čech compactification βN of the natural numbers N such that $A_1 \cup A_2 = \beta N$ and $A_1 \cap A_2 = N$; the product $A_1 \times A_2$ is not countably compact because it contains an infinite closed discrete space.

Additional conditions are thus necessary on one of the countably compact spaces X or Y to ensure countable compactness of the product $X \times Y$. Some of the additional properties on X which will guarantee this are: sequentially compact, first countable, sequential, k. These properties and some proofs have been discussed in a paper of S. Franklin [2]. Other properties which generate countably compact products in this manner are paracompactness and metacompactness, since either of these conditions, when added to countable compactness of a factor, makes the factor compact, and the product of a compact space with a countably compact space is well known to be countably compact. In what follows, assume all spaces Hausdorff.

A space which is a generalization of k-space (hence, of first countable and sequential space) has proved fruitful in some product theorems. This space is called a weakly-k space.

DEFINITION. A topological space X is weakly-k iff a subset F of X is closed in X if $F \cap C$ is finite for every compact C in X.

This notation was introduced in [5].

Weakly-k spaces can be used to generalize the results mentioned above.

THEOREM. *If X and Y are countably compact spaces such that X is weakly-k, then the product $X \times Y$ is countably compact.*

PROOF. Consider the sequence $\{(x_n, y_n): n \in N\}$ in $X \times Y$. If the sequence $\{x_n\}$ is closed, we are done, because there is a subsequence of $\{x_n\}$ with a limit point in X. So assume $x_n \neq x_m$ for all $m \neq n$ and that $\{x_n\}$ is not closed. By the weakly-k condition a compact C exists in X such that card $(C \cap \{x_n\})$

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$= \text{card}\{x_{n(k)}\} = \aleph_0$. Now $\text{cl}\{x_{n(k)}\} \subseteq C$, compact. Thus $\text{cl}\{x_{n(k)}\}$ is itself compact. On the other hand, $\{y_n\}$ is a sequence in Y which clusters, since Y is countably compact. So $\text{cl}\{x_{n(k)}\} \times \text{cl}\{y_{n(k)}\}$ is countably compact, hence any sequence in it clusters. Thus $\{(x_n, y_n)\}$ clusters, and $X \times Y$ is countably compact.

The above theorem then guarantees that Novák's spaces N_1 and N_2 are not weakly-k, since otherwise $N_1 \times N_2$ would be countably compact. They are also not k-spaces, so we ask whether an example exists of a weakly-k space which is not a k-space.

EXAMPLE. A weakly-k space which is not a k-space. Take $X = [0, \omega_0) \times [0, \Omega] \cup p$, with $p = \{\omega_0, \Omega\}$ and Ω the first uncountable ordinal. Let vertical fibers in X have the order topology. Also, let the point p have a neighborhood base consisting of sets of the form $[W \cap N \times (\alpha, \Omega)] \cup p$, where W is a neighborhood in βN of a fixed point in $\beta N - N$. In this way a topology is generated on X in which

(1) $N_1 = [0, \omega_0] \times \{\Omega\}$ is discrete,

(2) no sequence in N_1 converges to $\{\omega_0, \Omega\}$,

(3) $X - p$ is the sum of the vertical fibers.

Clearly $X - p$ is not closed. However, any compact C in X intersects only finitely many fibers, and a neighborhood of p can thus be found which does not intersect C. So $C \cap (X - p)$ is closed for every C in X, which proves that X is not a k-space. However, X is weakly-k, since whenever a set B is not closed, $B \cap C$ is infinite for some C compact contained in some fiber.

Other spaces, with varying degrees of strength or weakness, have been defined which ensure the productivity of countably compact spaces. See, for instance, the k_0-spaces of Chiba [1] and the $\mathfrak{C}^*$-spaces of Noble [3]. Both are equivalent to weakly-k spaces in the category of countably compact spaces.

A generalization of paracompact spaces which preserves products of countably compact spaces is the weakly-para-k space defined in [5]. The proof that this class preserves countable compactness of products is essentially the same as in the theorem above and will be omitted.

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