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ON THE PRODUCT AND COMPOSITION OF UNIVERSAL MAPPINGS OF MANIFOLDS INTO CUBES

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ABSTRACT. A map $f: X \to Y$ is said to be universal iff for every $g: X \to Y$ there exists $x \in X$ such that $f(x) = g(x)$.

Let M_t, $t \in T$, and M^n be orientable compact manifolds (in general with boundary). Let $\dim M^n = n$ and let Q_t be a cube with $\dim Q_t = \dim M_t$. Let $f_t: M_t \to Q_t, f_0: M^n \to I^n$ and $f_k: I^n \to I^n$ be universal mappings for $t \in T$ and $k = 1, 2, \ldots$. Then

(1.8) THEOREM. *The product map* $\prod_{t\in T} f_t: M_t \to \prod_{t\in T} Q_t$ *is universal.*

(2.1) THEOREM. *The composition* $f_s \circ f_{s-1} \circ \cdots \circ f_1: M^n \to I^n$ *is a universal map for* $s = 1, 2, \ldots$.

(2.2) THEOREM. *The limit X of the inverse sequence*

$$I^n \xleftarrow{f_1} I^n \xleftarrow{f_2} I^n \xleftarrow{f_3} \cdots$$

is an n-dimensional space with the fixed point property.

Some "counterexamples" are furnished. Also the following variant of Proposition (1.5) from [3] is given:

THEOREM A (PROPOSITION (1.5) OF [3]). *Let X be a compact space of (covering) dimension $\leqslant n$. Then $f: X \to I^n$ is a universal mapping iff the element $f^*(e^n)$ of the nth Čech cohomology group $H^n(X, f^{-1}(S^{n-1}); \mathbf{Z})$ is different from 0 for a generator e^n of $H^n(I^n, S^{n-1}; \mathbf{Z})$ where ($S^{n-1} = \partial I^n$).*

0. A mapping $f: X \to Y$ is said to be universal iff for every $g: X \to Y$ there exists $x \in X$ such that $f(x) = g(x)$. We will use the following results.

THEOREM A (PROPOSITION (1.5) OF [3]). *Let X be a compact space of (covering) dimension $\leqslant n$. Then $f: X \to I^n$ is a universal mapping iff the element $f^*(e^n)$ of the nth Čech cohomology group $H^n(X, f^{-1}(S^{n-1}); \mathbf{Z})$ is different from 0 for a generator e^n of $H^n(I^n, S^{n-1}; \mathbf{Z})$ (where $S^{n-1} = \partial I^n$).*

PROOF. Since $e^n = \delta s^{n-1}$ for some generator s^{n-1} of $H^{n-1}(S^{n-1})$ hence $f^*(e^n) = \delta f^*(s^{n-1})$. By Proposition (1.5) of [3], f is universal iff

$$\begin{aligned} f^*(s^{n-1}) &\notin \operatorname{Im}[H^{n-1}(X) \to H^{n-1}(f^{-1}(S^{n-1}))] \\ &= \operatorname{Ker}[\delta: H^{n-1}(f^{-1}(S^{n-1})) \to H^n(X, f^{-1}(S^{n-1}))], \end{aligned}$$

i.e., if $f^*(e^n) = \delta f^*(s^{n-1}) \neq 0$.

THEOREM B (SEE [2]). *Given a family of mappings of compact spaces, if the product of every finite subfamily is universal, then the product of all this family is a universal mapping.*

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THEOREM C (SEE [4]). *Given mappings* $f_k\colon X_{k-1} \to X_k$, $k = 1, 2, \ldots, n$. *If* $\prod_{k=1}^n f_k\colon \prod_{k=1}^n X_{k-1} \to \prod_{k=1}^n X_k$ *is universal, then the composition* $f_n \circ f_{n-1} \circ \cdots \circ f_1\colon X_1 \to X_n$ *is a universal mapping.*

THEOREM D (SEE [1]). *Let* $(X_t, p_t^u\colon t \leqslant u \in T)$ *be an inverse system of compact spaces* X_t *with the universal projections* p_t^u. *Then the limit of such a system has the fixed point property, if all* $X_t \in ANR$.

THEOREM E (SEE [5]). *Dimension of limit of an inverse system* $(X_t, p_t^u\colon t \leqslant u \in T)$ *of compact spaces, is* $\geqslant n$ *iff there exist an index* $t \in T$ *and a mapping* $f\colon X_t \to I_n$ *such that* $f \circ p_t^u\colon X_u \to I^n$ *is universal for every* $u \geqslant t$.

Our main tool will be Čech Cohomology and the Alexander-Pontriagin duality theorem (see Spanier [7]).

1. **Universality of product.**

(1.1) LEMMA. *Let* $f\colon M^n \to I^n$, $g\colon N^k \to I^k$ *be the universal mappings of n- and k-dimensional closed compact orientable manifolds* M^n *and* N^k. *Then* $f \times g\colon M^n \times N^k \to I^{n+k}$ *is a universal mapping.*

PROOF. Let $A = f^{-1}(S^{n-1})$, $B = g^{-1}(S^{k-1})$ and let

$$e^p \in H^p(I^p, S^{p-1}; Z)$$

be a generator. Then, by Theorem A,

$$(1.2) \quad 0 \neq f^*(e^n) \in H^n(M^n, A; Z) \quad \text{and} \quad 0 \neq g^*(e^k) \in H^k(N^k, B; Z).$$

By Theorem A, we must only prove that

$$(1.3) \quad 0 \neq (f \times g)^*(e^{n+k}) \in H^{n+k}(M^n \times N^k, (f \times g)^{-1}(S^{n+k-1}); Z).$$

But

$$\begin{aligned}(M^n \times N^k, (f \times g)^{-1}(S^{n+k-1})) &= (M^n \times N^k, A \times N^k \cup M^n \times B) \\ &= (M^n, A) \times (N^k, B)\end{aligned}$$

and

$$(1.4) \qquad (f \times g)^*(e^{n+k}) = f^*(e^n) \times g^*(e^k).$$

(We assume that the first of the following possible equalities holds

$$e^{n+k} = e^n \times e^k, \qquad e^{n+k} = -e^n \times e^k.)$$

By duality theorem the groups $H^n(M^n, A; Z)$ and $H^k(N^k, B; Z)$ have trivial torsion subgroups. Thus by Künneth formula, (1.3) holds.

Now, let for a mapping $f\colon M \to Y$, where M is a manifold, $f'\colon M' \to Y$ be the induced mapping of the double of M.

(1.5) PROPOSITION. *Mapping f is universal iff* f' *is universal.*

If also $g\colon N \to Z$ is a mapping of a manifold N, then

(1.6) PROPOSITION. *$f \times g$ is universal iff $f' \times g'$ is universal.*

PROOF OF PROPOSITIONS (1.5) AND (1.6). Let f_i: $M_i \to Y_i$ be a map from a manifold M into a space Y_i for $i = 1, \ldots, n$. Let f'_i: $M'_i \to Y_i$ be the induced map of the double M'_i of M_i. We may assume that $M_i \subseteq M'_i$ and that r_i: $M'_i \to M_i$ is the canonical retraction, $i = 1, \ldots, n$. Put $v = r_1 \times \cdots \times r_n$: $M'_1 \times \cdots \times M'_n \to M_1 \times \cdots \times M_n$ and

$$f = f_1 \times \cdots \times f_n \quad \text{and} \quad f' = f'_1 \times \cdots \times f'_n.$$

Then $f' = f \circ r$ and $f = f'|M_1 \times \cdots \times M_n$. Thus f' is universal iff f is. In the cases $n = 1$ and $n = 2$ we obtain the above two propositions.

(1.7) COROLLARY. *Let f: $M^n \to I^n$, g: $N^k \to I^k$ be the universal mappings of the compact n- and k-dimensional orientable manifolds M^n and N^k (in general with boundary). Then $f \times g$: $M^n \times N^k \to I^{n+k}$ is a universal mapping.*

It follows from the above corollary and Theorem B that the following theorem holds.

(1.8) THEOREM. *Product*

$$\prod_{t \in T} f_t\colon \prod_{t \in T} M_t \to \prod_{t \in T} Q_t$$

of a family of universal mappings $(f_t$: $M_t \to Q_t)_{t \in T}$ of orientable compact manifolds M_t (with boundary, in general) into cubes Q_t with dim $Q_t =$ dim $M_t < \infty$, *is a universal mapping.*

Now we shall show that the assumptions of Theorem (1.8) (and even of Lemma (1.1)) are essential.

(1.9) EXAMPLE. There is a universal mapping f: $M \to I^2$ of Möbius band and a universal mapping g: $I^2 \to I^2$ such that $f \times g$: $M \times I^2 \to I^4$ is not universal (see [4]).

(1.10) EXAMPLE. Klein bottle K is the double of M, and S^2 is the double of I^2. By Propositions (1.5) and (1.6), the mapping f': $K \to I^2$ and g': $S^2 \to I^2$ are universal (f and g are given as in Example (1.9)), but $f' \times g'$: $K \times S^2 \to I^4$ is not universal.

By a small modification it is easy to obtain two universal mappings of K into I^2 with nonuniversal product mapping of K^2 into I^4.

(1.11) EXAMPLE. It is remarked in [6], that there exists a universal mapping f: $I^4 \to I^3$ such that for identity 1_I: $I \to I$ (which is also a universal mapping) product $f \times 1_I$: $I^5 \to I^4$ is not universal.

Using quite elementary methods we can obtain the following generalization of Theorem (1.8).

(1.8′) THEOREM. *Let in Theorem* (1.8) *M_t be a compact space with arbitrary fine mappings onto orientable compact manifolds (instead of being a manifold), $t \in T$. Then under the other assumptions of Theorem* (1.8) *the assertion of this theorem holds.*

2. **An application to inverse limits of cubes.** As a corollary from Theorems C and (1.8) we obtain

(2.1) THEOREM. *If $f_k\colon I^n \to I^n$ is a universal mapping for $k = 1, 2, \ldots, s$, and $f_0\colon M^n \to I^n$ is a universal mapping of a compact orientable manifold M^n, then $f_s \circ f_{s-1} \circ \cdots \circ f_0\colon M^n \to I^n$ is a universal mapping.*

The following theorem is a direct consequence of Theorems D, E, and (2.1).

(2.2) THEOREM. *Let X be the limit of an inverse sequence*

$$I^n \xleftarrow{f_1} I^n \xleftarrow{f_2} I^n \xleftarrow{f_3} \cdots,$$

where $f_k\colon I^n \to I^n$ is a universal mapping for every $k = 1, 2, \ldots$. Then X is an n-dimensional space with the fixed point property.

3. **Examples of Cartesian squares.** In [4], there was mentioned an example of a universal map $f\colon B \to I^2$ of a 2-dimensional continuum B such that $f \times f\colon B \times B \to I^4$ is not universal (one takes B to be one of the Boltianski's continua). Since B is the limit of an inverse sequence of 2-dimensional polyhedra, hence applying Theorem E we obtain,

(3.1) THEOREM. *There exists a universal map $f\colon P \to I^2$ of a 2-dimensional connected polyhedron P such that $f \times f\colon P \times P \to I^4$ is not universal.*

Since every 2-dimensional polyhedron P is a retract of a 5-dimensional closed (i.e., compact and without boundary) orientable PL-manifold, it follows that

(3.2) THEOREM. *There exists a universal map $f\colon M \to I^2$ of a 5-dimensional closed orientable PL-manifold M such that $f \times f\colon M \times M \to I^4$ is not universal.*

The continuum B (see above) can be approximated by polyhedra embeddable even in $\mathbf{R}^4$. Thanks to this, the integer 5 in the above Theorem (3.2) can be replaced by arbitrary integer $\geqslant 4$.

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