PRINCIPAL CONGRUENCES OF $p$-ALGEBRAS
AND DOUBLE $p$-ALGEBRAS

T. HECHT AND T. KATRIŇÁK

Abstract. Principal congruence of pseudocomplemented lattices (= $p$-algebras) and of double pseudocomplemented lattices (= double $p$-algebras), i.e. pseudocomplemented and dual pseudocomplemented ones, are characterized.

1. Introduction. Recently H. Lakser [7] proved that every principal congruence of a distributive $p$-algebra is a join of two principal lattice congruences. We shall extend this result to all $p$-algebras (Theorem 1). The situation changes radically if one examines the double $p$-algebras. By Theorem 2, every principal congruence of a double $p$-algebra is a join of countably many principal lattice congruences. There exists even a distributive double $p$-algebra having a principal congruence which cannot be represented as a join of finite principal lattice congruences (Lemmas 3, 4 and Example). In Theorem 3 we give a necessary and sufficient condition in order that every principal congruence of a double $p$-algebra be a join of finite principal lattice congruences.

2. Preliminaries. A universal algebra $\langle L; \lor, \land, *, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ is called a $p$-algebra iff $\langle L; \lor, \land, 0, 1 \rangle$ is a bounded lattice such that for every $a \in L$ the element $a^* \in L$ is the pseudocomplement of $a$, i.e. $x \leq a^*$ iff $a \land x = 0$. A universal algebra $\langle L; \lor, \land, *, +, 0, 1 \rangle$ of type $\langle 2, 2, 1, 1, 0, 0 \rangle$ is called a double $p$-algebra iff $\langle L; \lor, \land, *, 0, 1 \rangle$ is a $p$-algebra and $\langle L; \lor, \land, +, 0, 1 \rangle$ is a dual $p$-algebra (x $\geq a^+$ iff $x \lor a = 1$). The standard results on $p$-algebras may be found in [3].

For a $p$-algebra $L$, define the set $B(L) = \{x \in L: x = x^{**}\}$ of closed elements. The partial ordering of $L$ partially orders $B(L)$ and makes the latter into a Boolean algebra $\langle B(L); \lor, \land, *, 0, 1 \rangle$ for which $a \lor b = (a \lor b)^{**}$ holds.

For any pair $a, b \in L$ in a $p$-, dual $p$-, or double $p$-algebra $L$, $\theta(a, b)$ denotes the principal congruence relation generated by $a, b$, i.e. the least congruence relation $\theta$ of this algebra for which $a \equiv b(\theta)$ is true. Clearly

$$\theta(a, b) = \theta(a \land b, a \lor b);$$

thus we need only characterize $\theta(a, b)$ for comparable $a, b$. We denote by

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\(\theta_{\text{Lat}}(a, b)\) the principal lattice congruence generated by \(a, b\); \(\theta_{\text{Lat}}(a, b)\) has the substitution property for \(\land\) and \(\lor\), but not necessarily for \(*\) or \(+\).

For the definition of a unary algebraic function see [2]. By a unary lattice function we mean such a unary algebraic function which can be obtained from a lattice polynomial (see also [3]).

3. Principal congruences of \(p\)-algebras.

**Lemma 1.** Let \(L\) be a \(p\)-algebra. Let \(a, b \in L\) and \(a \leq b\). If \(p(x)\) is a unary lattice function on \(L\) then the following identities hold:

(i) \(p(a)^* \land a^{**} = p(b)^* \land a^{**}\);

(ii) \(p(a)^* \land b^* = p(b)^* \land b^*\).

**Proof.** We proceed by induction on the rank of the lattice polynomial \(r(x_0, \ldots, x_{n-1})\), where \(r(x, c_1, \ldots, c_{n-1}) = p(x), c_1, \ldots, c_{n-1} \in L\). If \(p(x)\) is the identity or constant function (i) and (ii) hold trivially. Examine \(p(x) = q(x) \land t(x)\), \(q(x)\) and \(t(x)\) satisfying (i). Then

\[
p(a)^* \land a^{**} = [q(a) \land t(a)]^* \land a^{**} = (q(a)^* \land a^{**}) \cup (t(a)^* \land a^{**})
\]

\[
= (q(b)^* \land a^{**}) \cup (t(b)^* \land a^{**}) = p(b)^* \land a^{**}.
\]

Similarly, if \(p(x) = q(x) \lor t(x)\) and \(q(x)\) and \(t(x)\) satisfy (i), then

\[
p(a)^* \land a^{**} = [q(a) \lor t(a)]^* \land a^{**} = q(a)^* \land t(a)^* \land a^{**}
\]

\[
= q(b)^* \land t(b)^* \land a^{**} = p(b)^* \land a^{**}.
\]

Thus we have proved property (i). The proof of (ii) is similar.

**Lemma 2.** Let \(L\) be a lattice with 1 and let \(d \in L\). We define a binary relation \(\theta_d\) on \(L\) in the following way:

\[
x \equiv y(\theta_d) \quad \text{iff } x \land d = y \land d.
\]

Then \(\theta_d \leq \theta_{\text{Lat}}(d, 1)\).

The proof is straightforward.

**Theorem 1.** Let \(L\) be a \(p\)-algebra, let \(a, b \in L\) and let \(a \leq b\). Then

\[
(1) \quad \theta(a, b) = \theta_{\text{Lat}}(a, b) \lor \theta_{\text{Lat}}((a^* \land b)^*, 1).
\]

**Proof.** Let \(\theta\) denote the lattice congruence on the right-hand side of (1). First we show that \(\theta\) has the substitution property with respect to the operation \(.\). Let \(x \equiv y(\theta)\). Then there is a sequence \(x = z_0, \ldots, z_n = y\) of elements of \(L\) and a sequence \(p_0, \ldots, p_{n-1}\) of unary lattice functions such that

(I) \(\{z_i, z_{i+1}\} = \{p_i(a), p_i(b)\}\) or

(II) \(\{z_i, z_{i+1}\} = \{p_i((a^* \land b)^*), p_i(1)\}\) for any \(i = 0, 1, \ldots, n - 1\) holds.

Consider case (I). By Lemma 1, we have

\[
z_i^* \land a^{**} = z_{i+1}^* \land a^{**}, \quad z_i^* \land b^* = z_{i+1}^* \land b^*.
\]

Since \(B(L)\) is a Boolean algebra, we get
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The principal congruences of \( \langle a^{**} \cup b^{**} \rangle = z^{**}_{i+1} \land (a^{**} \cup b^{**}) \).

By Lemma 2, the last identity implies \( z^{**}_{i} = z^{**}_{i+1} (\theta_{\text{Lat}}((a^{*} \land b)^{*}, 1)) \), because \( a^{**} \cup b^{**} = (a^{*} \land b)^{*} \). In case (II) we obtain

\[ z^{*}_{i} \land (a^{*} \land b)^{*} = z^{*+1}_{i} \land (a^{*} \land b)^{*} \]

by Lemma 1(i), bearing in mind \( (a^{*} \land b)^{*} \in B(L) \). This implies \( z^{*}_{i} = z^{*+1}_{i} (\theta_{\text{Lat}}((a^{*} \land b)^{*}, 1)) \) by Lemma 2. So, \( \chi^{*} = \chi^{*} (\theta) \) and \( \theta \) is a \( * \)-congruence of \( L \). Evidently \( \theta(a, b) \leq \theta \). Conversely, \( a = b(\theta(a, b)) \) yields \( a^{*} \land b = 0(\theta(a, b)) \), and hence \( (a^{*} \land b)^{*} = 1(\theta(a, b)) \). Thus, \( \theta(a, b) \geq \theta \). Concluding, \( \theta(a, b) = \theta \).

**Corollary 1.** Let \( L \) be a \( p \)-algebra. Then \( \theta(a, 1) = \theta_{\text{Lat}}(a, 1) \) for every \( a \in L \).

**Corollary 2.** Let \( L \) be a dual \( p \)-algebra, let \( a, b \in L \) and let \( a \leq b \). Then

\[ \theta(a, b) = \theta_{\text{Lat}}(a, b) \lor \theta_{\text{Lat}}((a \lor b^{*})^{*}, 0). \]

**Corollary 3.** Let \( L \) be a dual \( p \)-algebra. Then \( \theta(0, a) = \theta_{\text{Lat}}(0, a) \) for every \( a \in L \).

**Remark 1.** The analogue of Theorem 1 is also valid for the pseudocomplemented semilattices. (The proof of Theorem 1 is based on the fact that \( B(L) \) is a Boolean algebra.)

**Remark 2.** Theorem 1 was proved in [7] for the distributive \( p \)-algebras. In [4], an equivalent version of Theorem 1 has been proved for the modular \( S \)-algebras.

4. Principal congruences of double \( p \)-algebras. Let \( L \) be a double \( p \)-algebra, let \( x \in L \). We define \( x^{n(++)} \in L \) in the following way: \( x^{1(++)} = x^{++} \), \( x^{k+1(++)} = x^{k(++)} \lor x^{k(++)} \) for every \( k \geq 1 \). Similarly we define \( x^{m(++)} \in L \). Since \( a^{*} \lor a^{*} = 1 \) implies \( a^{**} \land a^{*} = 0 \), we obtain \( a^{**} \leq a^{*} \) in \( L \). Therefore,

\[ x^{*} \geq x^{**} \geq \ldots \geq x^{m(++)} \geq \ldots \]

in \( L \). Dually we have

\[ y^{+} \leq y^{++} \leq \ldots \leq y^{m(++)} \leq \ldots \]

for any \( y \in L \).

**Theorem 2.** Let \( L \) be a double \( p \)-algebra, let \( a, b \in L \) and let \( a \leq b \). Then

\[ \theta(a, b) = \theta_{\text{Lat}}(a, b) \]

\[ \lor \lor_{n \geq 0} \left[ \theta_{\text{Lat}}((a^{*} \land b)^{n(++)}, 1) \lor \theta_{\text{Lat}}(0, (a \lor b^{*})^{n(++)}) \right]. \]

**Proof.** Let \( \theta \) denote the lattice congruence on the right-hand side of (5). It is a routine to show that \( \theta(a, b) \geq \theta \). To conclude the proof we need only to show that \( \theta \) has the substitution property with respect to the operations \( * \) and
First we prove that $\theta$ is a *-congruence. Let $x = y(\theta)$. Then there is a sequence $x = z_0, \ldots, z_n = y$ of elements of $L$ and congruences $\theta_0, \ldots, \theta_{n-1}$ such that $z_i = z_{i+1}(\theta_i)$ where

1. $\theta_i = \theta_{\text{Lat}}(a, b)$ or
2. $\theta_i = \theta_{\text{Lat}}((a^* \land b)^k(\ldots), 1)$ for some $k \geq 0$ or
3. $\theta_i = \theta_{\text{Lat}}(0, (a \lor b^\dagger)^m(\ldots))$ for some $m \geq 0$, and for any $i = 0, \ldots, n - 1$.

1. $z_i \equiv z_{i+1}(\theta_{\text{Lat}}(a, b))$ implies

$$z_i = z_{i+1}(\theta_{\text{Lat}}(a, b) \lor \theta_{\text{Lat}}((a^* \land b)^*, 1)),$$

by Theorem 1. Therefore $z_i \equiv z_{i+1}(\theta)$.

2. $z_i \equiv z_{i+1}(\theta_{\text{Lat}}((a^* \land b)^k(\ldots), 1))$ implies

$$z_i \equiv z_{i+1}(\theta_{\text{Lat}}((a^* \land b)^k(\ldots), 1)),$$

by Corollary 1 to Theorem 1. Therefore $z_i \equiv z_{i+1}(\theta)$.

3. $z_i \equiv z_{i+1}(\theta_{\text{Lat}}(0, (a \lor b^\dagger)^m(\ldots)))$ implies

$$z_i \equiv z_{i+1}(\theta_{\text{Lat}}(0, (a \lor b^\dagger)^{m+1}(\ldots)) \lor \theta_{\text{Lat}}((a \lor b^\dagger)^{m+1}(\ldots), 1)),$$

by Theorem 1. Since $(a \lor b^\dagger)^{m+1}(\ldots) \lor (a \lor b^\dagger)^{m+1}(\ldots) = 1$, we have

$$\theta_{\text{Lat}}(0, (a \lor b^\dagger)^{m+1}(\ldots)) = \theta_{\text{Lat}}((a \lor b^\dagger)^{m+1}(\ldots), 1).$$

Therefore $z_i \equiv z_{i+1}(\theta)$.

Thus, $z_i \equiv z_{i+1}(\theta)$ for any $i = 0, \ldots, n - 1$, and we have proved $x^* \equiv y^*(\theta)$, i.e. $\theta$ is a *-congruence of $L$. Using Corollaries 2 and 3 to Theorem 1 one can similarly prove that $\theta$ is a +-congruence of $L$, and so the proof is complete.

**Corollary 1.** Let $L$ be a double $p$-algebra, $a \in L$. Then

(6) $\theta(a, 1) = \lor_{n \geq 0} \theta_{\text{Lat}}(0, a^{n\dagger}(\ldots))$;

(7) $\theta(0, a) = \lor_{n \geq 0} \theta_{\text{Lat}}(a^{n\dagger}(\ldots), 1)$.

**Proof.** We know that $\theta_{\text{Lat}}(a, 1) \leq \theta_{\text{Lat}}(0, a^\dagger)$ and $\theta_{\text{Lat}}(a^{m\dagger}(\ldots), 1) \leq \theta_{\text{Lat}}(0, a^{m\dagger}(\ldots))$ is true. Hence, by Theorem 2, we have (6). By dual arguments we can prove (7).

**Corollary 2.** Let $L$ be a double $p$-algebra in which the chains (3) and (4) are finite for every $x, y \in L$. Let $a, b \in L$ with $a \leq b$. Let $m$ be the least number with the property $(a^* \land b)^{m\dagger}(\ldots) = (a^* \land b)^{m+1}(\ldots)$ and $(a \lor b^\dagger)^{-m(\ldots)} = (a \lor b^\dagger)^{(m+1)(\ldots)}$. Then

$$\theta(a, b) = \theta_{\text{Lat}}(a, b) \lor \theta_{\text{Lat}}((a^* \land b)^{m\dagger}(\ldots), 1) \lor \theta_{\text{Lat}}(0, (a \lor b^\dagger)^{(m+1)(\ldots)}).$$
Corollary 3. Let $L$ be a distributive double $p$-algebra satisfying the identities

$$x^{*m(++)} = x^{*(m+1)(+++)}$$
$$x^{+m(++)} = x^{+(m+1)(+++)}$$

for some $m \geq 0$. Let $a, b \in L$ with $a \leq b$. If $x, y \in L$, then $x \equiv y(\theta(a, b))$ iff

$$[x \wedge a \wedge (a^* \wedge b)^{m(++)}] \lor (a \lor b^+)^{+m(++)}$$
$$= [y \wedge a \wedge (a^* \wedge b)^{m(++)}] \lor (a \lor b^+)^{+m(++)}$$

and

$$[(x \lor b) \wedge (a^* \wedge b)^{m(++)}] \lor (a \lor b^+)^{+m(++)}$$
$$= [(y \lor b) \wedge (a^* \wedge b)^{m(++)}] \lor (a \lor b^+)^{+m(++)}.$$

Proof. The proof follows from Corollary 2 and the fact that in a bounded distributive lattice $L$, for any elements $a_1, b_1, a_2, b_2 \in L$ with $a_1 \leq b_1$, the following statement is true:

$$x = y(\theta_{\text{Lat}}(a_1, b_1) \lor \theta_{\text{Lat}}(a_2, 1) \lor \theta_{\text{Lat}}(0, b_2))$$

iff

$$(x \wedge a_1 \wedge a_2) \lor b_2 = (y \wedge a_1 \wedge a_2) \lor b_2$$

and

$$[(x \lor b_1) \wedge a_2] \lor b_2 = [(y \lor b_1) \wedge a_2] \lor b_2$$

hold (see [3]).

Remark. Corollary 3 combined with the result of A. Day [1] says that the equational subclass of the class of all distributive double $p$-algebras determined by the identities from Corollary 3 enjoys the Congruence Extension Property. We note here that the whole class of distributive double $p$-algebras has CEP (see [5]). Corollary 3 solves partially the problem mentioned in [5].

5. Counterexample. In this part we shall construct a distributive double $p$-algebra having a principal congruence which cannot be represented as a join of finite principal lattice congruences.

Lemma 3. Let $L$ be a double $p$-algebra. If $a \in L$ and $a^{*n(++)} \geq a^{*(n+1)(++)}$ for every integer $n \geq 0$ then $\theta(0, a)$ cannot be represented as a join of finite principal lattice congruences of $L$.

Proof. Let $a^{*n(++)} \geq a^{*(n+1)(++)}$ for any $n \geq 0$. Suppose to the contrary that $\theta(0, a)$ is a join of finite principal lattice congruences of $L$. Then $\theta(0, a)$ is a compact element of the lattice of all lattice congruences on $L$ (cf. [2]). Therefore, by (3) and (6), there exists an integer $k \geq 0$ such that

$$\theta(0, a) = \bigvee_{n=0}^{k} \theta_{\text{Lat}}(a^{*n(++)}, 1) = \theta_{\text{Lat}}(a^{*k(++)}, 1).$$
Evidently $a^{*(k+1)(+*)} = 1(\theta(0, a))$. On the other hand,
$$a^{*(k+1)(+*)} \neq 1(\theta_{\text{Lat}}(a^{*k(+*)}, 1))$$
(see [3]), a contradiction. The second part can be proved dually.

**Lemma 4.** Let $B$ be a Boolean algebra, let $\varphi: B \to B$ be a $\{0, 1, \wedge\}$-homomorphism and let $\psi: B \to B$ be a $\{0, 1, \vee\}$-homomorphism such that $a\varphi \varphi \leq a$ and $a\psi \varphi \geq a$ for every $a \in B$. Then the set $L = \{(a, b) \in B^2: a\varphi \geq b\}$ is a $\{0, 1\}$-sublattice of $B^2$ and, moreover, $L$ forms a distributive double $\varphi$-algebra in which for $t = (a, b) \in L$,
$$t^* = (a', a'^\varphi), \quad t^+ = (b', b')$$
is true.

For the proof see [6, Theorem 2].

**Example.** Let $B$ denote the Boolean algebra of all subsets of the set $N$ of positive integers. Set
$$A\varphi = \{x \in N: x \in A \text{ and } x + 1 \in A\} \text{ for every } A \in B,$$
$$A\psi = \{x \in N: x \in A \text{ or } x - 1 \in A\} \text{ for every } A \in B.$$ It is easy to verify that $\varphi$ is a $\{0, 1, \wedge\}$-homomorphism of $B$ into $B$, $\psi$ is a $\{0, 1, \vee\}$-homomorphism of $B$ into $B$, both of which satisfy $A\varphi \varphi \leq A$ and $A\psi \varphi \geq A$ for every $A \in B$. Let $L = \{(X, Y) \in B^2: X\varphi \geq Y\}$ be the distributive double $\varphi$-algebra (see Lemma 4). Let $K_n = \{1, \ldots, n\} \in B$. If we set $a = (N - K_1, N - K_1)$ and $b = a^*$ then $a^* = b = (K_2, K_1)$, $a^{**} = (N - K_2, N - K_2)$, $a^{+++} = b^* = (K_3, K_2)$. By induction it is easy to prove
$$a^{+n(++)} = (K_{n+2}, K_{n+1}),$$
$$b^{+n(++)} = (N - K_{n+2}, N - K_{n+2}).$$
Now we see that $b^{+n(++)} > b^{+n+1(++)}$ and $a^{+n(++)} < a^{+n+1(++)}$ for every integer $n \geq 0$. So, by Lemma 3, $\theta(a, 1)$ and $\theta(0, b)$ cannot be represented as a join of finite principal lattice congruences of $L$.

Concluding we obtain

**Theorem 3.** Let $L$ be a double $\varphi$-algebra. Let $a, b \in L$. Then the principal congruence $\theta(a, b)$ is a join of finite principal lattice congruences of $L$ iff the chains (3) and (4) are finite for every $x, y \in L$.

The proof follows from Corollary 2 to Theorem 2 and Lemma 3.

**References**


Katedra Algébry a Teória Čísel PFUK, 81631 Bratislava 16, Mlynská Dolina, Czechoslovakia