PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, July 1976

LIE *-TRIPLE HOMOMORPHISMS INTO VON NEUMANN ALGEBRAS

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ABSTRACT. Let M and N be associative *-algebras. A Lie *-triple homomorphism of M into N is a *-linear map $\phi: M \to N$ such that

$$\phi[[A, B], C] = [[\phi(A), \phi(B)], \phi(C)].$$

(Here M and N are considered as Lie *-algebras with $[X, Y] = XY - YX$.) In this note we prove that if N is a von Neumann algebra with no central abelian projections and if ϕ is onto, there exists a central projection D in N such that $D\phi$ is a Lie *-homomorphism of $[M, M]$, and $(I - D)\phi$ is a Lie *-antihomomorphism of $[M, M]$.

1. **Introduction.** An associative algebra M can be turned into a Lie algebra by defining a new multiplication $[X, Y] = XY - YX$ where XY is the associative product of X and Y. Every abstract Lie algebra is isomorphic to a subalgebra of a Lie algebra formed in this way. A Lie triple system is a subspace of M closed under the Lie triple product $[[A, B], C]$. Lie triple systems and their homomorphisms have been studied in relation to Jordan homomorphisms of rings and the following theorem proved [2, Theorem 15]:

Let ϕ be a Lie triple system homomorphism of the special Lie ring L and denote by M the enveloping Lie ring of $\phi(L)$ and Z the centre of M. Assume (i) M/Z has no commutative Lie ideals and (ii) any two nonzero Lie ideals in M/Z have nonzero intersection. Then ϕ, when restricted to the Lie ring $[L, L]$, is either a Lie homomorphism or antihomomorphism.

We wish to prove an analogous theorem when the image algebra is a von Neumann algebra. The situation is complicated by the presence, in the general case, of nonzero central projections which makes (ii) of the above theorem inapplicable.

2. **Notation and preliminaries.** M is a *-algebra over the complex field and M_0, M_1 subsets of M, then $[M_0, M_1]$ = all finite linear combinations of elements of the form $[A, B]$ with $A \in M_0$, $B \in M_1$. A Lie *-triple homomorphism $\phi: M \to N$ is a *-linear map preserving the Lie triple product $[[A, B], C]$. The enveloping Lie algebra [2, p. 493] of $\phi(M)$ is the set $\phi(M) + [\phi(M), \phi(M)]$. A Lie *-ideal of M is a *-linear subspace $U \subseteq M$ such that if $Y \in U$, $[X, Y] \in U$ for all $X \in M$.

A von Neumann algebra M is a weakly closed, selfadjoint algebra of operators on a complex Hilbert space H containing the identity operator I. The set $Z_M = \{S \in M: [S, T] = 0 \text{ for all } T \in M\}$ is called the centre of M.

Received by the editors September 30, 1975.
AMS (MOS) subject classifications (1970). Primary 46L10; Secondary 16A68.
Key words and phrases. Lie *-triple homomorphism, von Neumann algebra.

If P is a projection (= selfadjoint idempotent) in M, then $M_P = \{PAP | A \in M\}$. A projection P is abelian if M_P is an abelian algebra. We use [1] as a general reference for the theory of von Neumann algebras.

The following fact will be used several times in what follows: If M is a C^*-algebra, $X, Y \in M$ with $Y = Y^*$, then $[X, Y] \in Z_M$ implies $[X, Y] = 0$ [2, Lemma 6]. This implies, for example, that if M_0 and M_1 are subsets of M with M_1 a *-subspace, then $[M_0, M_1] \subseteq Z_M$ implies $[M_0, M_1] = \{0\}$.

3. **Lie *-triple homomorphisms.** Let $\phi: M \to N$ be a Lie *-triple homomorphism where M is a *-algebra over $\mathbf{C}$ and N is a von Neumann algebra. The case where N is a factor (that is $Z_N = \{\lambda I: \lambda \in \mathbf{C}\}$) is included separately, even though the factor case fits into the general theorem, since ϕ can be analyzed when N is a factor by using the Jacobson-Rickart theorem already mentioned. The following result may be of independent interest.

LEMMA 1. *Let N be a C^*-algebra. Then N/Z_N, considered as a Lie *-algebra, contains no nontrivial abelian Lie *-ideals.*

PROOF. Let N_0 be an abelian Lie *-ideal in N/Z_N and let $\pi: N \to N/Z_N$ be the canonical Lie *-homomorphism where $\pi(A) = A + Z_N$. N_0 is generated, as a *-linear space, by selfadjoint elements so let $A + Z_N$, $B + Z_N$ be elements of N_0 with $A - A^* \in Z_N$ and $B - B^* \in Z_N$. Then $\pi([A, B]) = [\pi(A), \pi(B)] = 0$ since N_0 is abelian. Thus $[A, B] \in \ker \pi = Z_N$. Now $A - A^* \in Z_N$ implies $[A^*, B] = [A, B] \in Z_N$ so that $[A + A^*, B] \in Z_N$. This forces $[A + A^*, B] = 0$. Similarly $[A - A^*, B] = 0$. Adding, we have $[A, B] = 0$. $\pi^{-1}(N_0) + Z_N$ is therefore an abelian Lie *-ideal in N so that by [3, Lemma 36], $\pi^{-1}(N_0) \subseteq Z_N$ or $N_0 = \{0\}$.

THEOREM 1. *If N is a factor and $\phi: M \to N$ is a Lie *-triple homomorphism of M onto N then $\phi|_{[M,M]}$ is a Lie *-homomorphism or a Lie *-antihomomorphism.*

PROOF. Since ϕ is onto, $\phi(M) + [\phi(M), \phi(M)] = N$ so that we need only show condition (ii) of the Jacobson-Rickart theorem is fulfilled. Let U_1 and U_2 be nonzero Lie *-ideals in N/Z_N and let $V_1 = \pi^{-1}(U_1)$, $V_2 = \pi^{-1}(U_2)$. Then $V_1 + Z_N$, $V_2 + Z_N$ are Lie *-ideals in N and neither is contained in Z_N.

By [3, Lemma 37] there exist nonzero two-sided ideals $\mathcal{I}_1$, $\mathcal{I}_2$ of N such that $[\mathcal{I}_1, N] \subseteq V_1 + Z_N$ and $[\mathcal{I}_2, N] \subseteq V_2 + Z_N$. If $[\mathcal{I}_1, N] \subseteq Z_N$ then $[\mathcal{I}_1, N] = 0$ and $\mathcal{I}_1 \subseteq Z_N = \{\lambda I | \lambda \in \mathbf{C}\}$ which would force $\mathcal{I}_1 = \{0\}$. If $\mathcal{I}_1 = N$ then

$$[\mathcal{I}_1, N] \cap [\mathcal{I}_2, N] = [\mathcal{I}_2, N] \subseteq (V_1 + Z_N) \cap (V_2 + Z_N)$$

so that $U_1 \cap U_2 \neq \{0\}$.

So we can assume $U_1 \cap U_2 = \{0\}$ and $\mathcal{I}_1$, $\mathcal{I}_2$ are nonzero, proper ideals in N. Now

$$\pi^{-1}(\{0\}) = \pi^{-1}(U_1 \cap U_2) = (V_1 + Z_N) \cap (V_2 \cap Z_N) \subseteq Z_N.$$

Hence $[\mathcal{I}_1, N] \cap [\mathcal{I}_2, N] \subseteq Z_N$ which implies $[\mathcal{I}_1, \mathcal{I}_2] \subseteq Z_N$. Since $V_1 + Z_N$ and $V_2 + Z_N$ are selfadjoint collections, we can assume the same of $\mathcal{I}_1$ and $\mathcal{I}_2$ so that $[\mathcal{I}_1, \mathcal{I}_2] = \{0\}$. Moreover $[\mathcal{I}_1 \mathcal{I}_2, N] \subseteq [\mathcal{I}_1, N] \cap [\mathcal{I}_2, N] \subseteq Z_N$ so that $[\mathcal{I}_1 \mathcal{I}_2, N] = \{0\}$. Hence $\mathcal{I}_1 \mathcal{I}_2$ is a selfadjoint two-sided ideal in Z_N so that $\mathcal{J}_1 \mathcal{J}_2 = \{0\}$ which is impossible since N is a factor.

We now turn our attention to the general case. As in [2] the sets

$$N^+ = \left\{ \sum_{i=1}^{n} \phi[A_i, B_i] - [\phi(A_i), \phi(B_i)]: A_i, B_i \in M \right\}$$

and

$$N^- = \left\{ \sum_{i=1}^{n} \phi[A_i, B_i] - [\phi(B_i), \phi(A_i)]: A_i, B_i \in M \right\}$$

are Lie ideals in $\phi(M) + [\phi(M), \phi(M)]$. In our case N^+ and N^- are also closed under the *-operation since ϕ preserves adjoints. If, for example, $N^+ \subseteq Z_N$ then

$$\begin{aligned} 0 &= [\phi[A, B] - [\phi(A), \phi(B)], \phi[X, Y]] \\ &= [\phi[A, B], \phi[X, Y]] - \phi[[A, B], [X, Y]] \end{aligned}$$

so that ϕ is a Lie *-homomorphism of $[M, M]$. Similarly, if $N^- \subseteq Z_N$ then ϕ is a Lie *-antihomomorphism of $[M, M]$.

LEMMA 2. *Let ϕ: $M \to N$ be a Lie *-triple homomorphism of the *-algebra M onto a von Neumann algebra N which has no abelian central projections and suppose $N^+ \not\subseteq Z_N$ and $N^- \not\subseteq Z_N$. There exist projections $C \neq 0$ and $D \neq 0$ in Z_N such that $N^+ + Z_N \subseteq N_C + Z_N$, $N^- + Z_N \subseteq N_D + Z_N$ and $CD = 0$.*

PROOF. By [2, Theorem 14] we have $[N^+, N^-] \subseteq Z_N$ and so $[N^+, N^-] = 0$, since N^+, N^- are selfadjoint collections. Hence N^+, N^- are commuting Lie *-ideals so that $(N^+ + Z_N)^{-uw}$ and $(N^- + Z_N)^{-uw}$ are also commuting Lie *-ideals. ($(N^+ + Z_N)^{-uw}$ is the ultra-weak closure of $(N^+ + Z_N)$.) By [3, Theorem 4, Corollary], $(N^+ + Z_N)^{-uw} = N_C + Z_N$, $(N^- + Z_N)^{-uw} = N_D + Z_N$ where $C \neq 0$, $D \neq 0$ are projections in Z_N. Since these Lie *-ideals commute we have $[N_C, N_D] = [N_{CD}, N_{CD}] = 0$ or CD is a central abelian projection. Thus $CD = 0$.

THEOREM 2. *Let $\phi: M \to N$ be a Lie *-triple homomorphism of a *-algebra M onto a von Neumann algebra N which has no central abelian projections. There exists a projection $D \in Z_N$ such that $D\phi$ is a Lie *-homomorphism on $[M, M]$ and $(I - D)\phi$ is a Lie *-antihomomorphism on $[M, M]$.*

PROOF. If $N^+ \subseteq Z_N$ or $N^- \subseteq Z_N$ then $D = 0$ or $D = I$. Otherwise there exist projections $C \neq 0$, $D \neq 0$ in Z_N such that $N^+ + Z_N \subseteq N_C + Z_N$, $N^- + Z_N \subseteq N_D + Z_N$ and $CD = 0$. We have $N^+D = \{TD | T \in N^+\} \subseteq Z_N D$ and $N^- C \subseteq Z_N C$. By the discussion before Lemma 2 we have that $D\phi$ is a Lie *-homomorphism of $[M, M]$ and $C\phi$ is a Lie *-antihomomorphism of $[M, M]$.

Now $N^+(I - C - D) \subseteq Z_N(I - C - D)$ and $N^-(I - C - D) \subseteq Z_N(I - C - D)$ so that $(I - C - D)\phi$ is both a Lie *-homomorphism and a Lie *-antihomomorphism of $[M, M]$. Thus if $X, Y \in [M, M]$,

$$\begin{aligned} (I - C - D)\phi[X, Y] &= (I - C - D)[\phi(X), \phi(Y)] \\ &= (I - C - D)[\phi(Y), \phi(X)]. \end{aligned}$$

This implies $(I - C - D)\,\phi(X)\,\phi(Y) = (I - C - D)\,\phi(Y)\,\phi(X)$ or that $(I - C - D)\,\phi\,[M, M]$ is abelian. $[[M, M], M] \subseteq [M, M]$ so that

$$\left[\left[\phi(M), \phi(M)\right], \phi(M)\right] \subseteq \phi\left[M, M\right].$$

Since ϕ is onto, $[[N, N], N] \subseteq \phi[M, M]$ and

$$\left[\left[N_{(I-C-D)}, N_{(I-C-D)}\right], N_{I-C-D}\right] \subseteq (I - C - D)\phi\left[M, M\right].$$

Hence $[[N_{(I-C-D)}, N_{I-C-D}], N_{I-C-D}]$ is an abelian Lie *-ideal in N and contained in Z_N. This implies $N_{(I-C-D)}$ is abelian so that $I - C - D = 0$.

REMARK 1. The requirement that ϕ be onto is made so that N^+ and N^-, which are Lie *-ideals of $\phi(M) + [\phi(M), \phi(M)]$ will be Lie *-ideals in N where a characterization of such ideals is known. Other restrictions on M, N and ϕ can be made to insure that $\phi(M) + [\phi(M), \phi(M)] = N$. ϕ is called L-onto if, given $Y \in N$, there exists $X \in M$ such that $\phi(X) - Y \in Z_N$.

If N is an infinite von Neumann allebra then $[N, N] = N$ [5, Theorem 2]. Hence if ϕ is L-onto and N is infinite, $[\phi(M), \phi(M)] = [N, N] = N$. If N is a type I finite von Neumann algebra then $Z_N + [N, N] = N$ [4, Theorem 1]. If in this case ϕ were L-onto and $Z_N \subseteq \phi(M)$, we would have $N = Z_N + [N, N] \subseteq \phi(M) + [\phi(M), \phi(M)] \subseteq N$.

REMARK 2. Modification of the arguments of [3] shows that if M and N are von Neumann algebras with no central abelian projections and ϕ is L-onto, then Z_M and Z_N are *-isomorphic.

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