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ON GENERALIZED RESOLVENTS

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ABSTRACT. Let T be a bounded linear operator on a Hilbert space and $\rho_I(T)$ the Fredholm domain of T. It is shown that a generalized resolvent can be constructed for T in $\rho_F(T)$ which verifies the resolvent identity except for an at most countable set of points which are close to the boundary of $\rho_F(T)$.

Let T be a bounded linear operator on a Hilbert space H. In case the range of T is a closed subspace of H, then an operator F will be called a generalized inverse of T when FT is a projection onto the orthogonal complement of the kernel of T and TF is a projection onto the range of T. Unless T is invertible, then a generalized inverse is not unique. Let $\mathcal{G}$ be a domain in the complex plane $\mathbf{C}$ such that for every λ in $\mathcal{G}$, the operator $\lambda - T$ has closed range. An operator valued function F defined on $\mathcal{G}$ is called a generalized inverse function for T in $\mathcal{G}$ in case, for every λ in $\mathcal{G}$, $F(\lambda)$ is a generalized inverse of $\lambda - T$. A generalized inverse function F for T on an open set $\mathcal{G}$ is said to verify the resolvent identity on $\mathcal{G}$, when for every pair λ, μ in a component of $\mathcal{G}$

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu). \tag{1}$$

A continuous generalized inverse function, for an operator T on an open set $\mathcal{G}$, which verifies the resolvent identity on $\mathcal{G}$ will be called a generalized resolvent on $\mathcal{G}$.

This note is concerned with the construction of generalized resolvents on open subsets of the Fredholm domain of a bounded operator T. Recall that an operator T is called semi-Fredholm in case T has closed range and the dimension of at least one of $\ker(T)$ or $\ker(T^*)$ is finite; here, ker denotes kernel and T^* is the adjoint of T. If T has closed range and both $\ker(T)$ and $\ker(T^*)$ are finite dimensional, then T is called a Fredholm operator. The semi-Fredholm domain of T is the set $\rho_{s-F}(T) = \{\lambda \in \mathbf{C}\colon \lambda - T$ is semi-Fredholm$\}$ and the Fredholm domain of T is the set $\rho_F(T) = \{\lambda \in \mathbf{C}\colon \lambda - T$ is Fredholm$\}$.

There is one obvious obstruction to constructing a generalized resolvent for T on all of $\rho_{s-F}(T)$. In $\rho_{s-F}(T)$ there is an at most countable set where the function

$$m(\lambda) = \text{minimum dimension}\left[\ker(\lambda - T), \ker(\lambda - T)^*\right]$$

is discontinuous [3, Proposition 2.6], [5], [6]. This set will be denoted by $\rho_{s-F}^s(T)$ and is referred to as the set of singular points in the semi-Fredholm

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domain. The singular points in the Fredholm domain $\rho_{s-F}^{s}(T) \cap \rho_F(T)$ will be denoted by $\rho_F^s(T)$. The set $\rho_{s-F}^{s}(T)$ does not accumulate in $\rho_{s-F}(T)$ (see, e.g., [3], [5], [6]) and it is easy to show $\rho_r(T) \cap \rho_{s-F}^{s}(T) = \varnothing$. The complementary set $\rho_{s-F}^{r}(T) = \rho_{s-F}(T) \setminus \rho_{s-F}^{s}(T)$ ($\rho_F^r(T) = \rho_F(T) \setminus \rho_F^s(T)$) is called the set of regular points in the semi-Fredholm (Fredholm) domain. Obviously, there does not exist a continuous generalized inverse function in a neighborhood of a point $\lambda \in \rho_{s-F}^{s}(T)$.

The notation bdry $\mathcal{G}$ will be used for the boundary of a subset of $\mathbf{C}$ and dist(S, S') will denote the Hausdorff distance between two bounded sets S, S' in $\mathbf{C}$. In other words,

$$\operatorname{dist}(S, S') = \max\left[\sup_{\lambda \in S'}\left[\operatorname{distance}(\lambda, S)\right], \sup_{\lambda \in S}\left[\operatorname{distance}(\lambda, S')\right]\right].$$

The main result to be established here is

THEOREM 1. *Let T be a bounded operator on H and let $\varepsilon > 0$. There exists a generalized resolvent on $\rho_F^r(T)$ except for an at most countable set S which does not accumulate in $\rho_F(T)$. Moreover,* dist$(S$, bdry $\rho_F(T)) < \varepsilon$.

There are several papers in the literature which contain results similar in spirit to the above theorem. In [8] P. Saphar obtains the above theorem (in the generality of operators on a Banach space) with the conclusion

"dist$(S$, bdry $\rho_F(T)) < \varepsilon$"

replaced by

"dist$(S$, bdry $\rho_F^r(T)) < \varepsilon$".

Also Shapiro and Schechter [9] construct generalized resolvents on $\rho_F^r(T)$, minus a countable set S for operators T acting on a Banach space. These authors do not make any attempt to push the set S out near the boundary of $\rho_F(T)$.

It is clear that a generalized resolvent for T defined on an open set $\mathcal{G}$ is an analytic generalized inverse function for T in $\mathcal{G}$. On the other hand, not every analytic generalized inverse function defined on an open set $\mathcal{G}$ verifies the resolvent identity on $\mathcal{G}$. Let $\rho_r(T)$ ($\rho_l(T)$) denote the set of complex λ, where $\lambda - T$ has a right (left) inverse. Allan [1], [2] has shown that there exists an analytic right (left) inverse function for T in $\rho_r(T)$ ($\rho_l(T)$). This fact can be used to construct an analytic generalized inverse function for an operator T in $\rho_{s-F}^{r}(T)$ [4].

Using the result in Theorem 1 it is possible to construct a "generalized spectral projection" associated with any finite subset σ of $\rho_F^s(T)$. This leads to a decomposition of the operator T as a direct sum $T_1 \oplus T_2$ (not necessarily an orthogonal sum), where T_1 contains the singular points σ as isolated eigenvalues in its spectrum and the operator T_2 satisfies $\rho_F^r(T_2) = \rho_F^r(T) \cup \sigma$. The reader is referred to [4] for further details.

1. **Preliminaries**. This section begins with a few basic lemmas:

LEMMA 1. *Let T be an operator on H and let H_1, H_2 be closed subspaces of H. Assume that relative to the orthogonal decomposition $H = H_1 \oplus H_2$ the* *operator T has the 2×2 matrix representation*

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Let $\mathcal{G}$ be an open set on which both A, B have right resolvents R_1, R_2 respectively. Then

$$R = \begin{pmatrix} R_1 & R_1 C R_2 \\ 0 & R_2 \end{pmatrix}$$

defines a right resolvent of T on $\mathcal{G}$.

PROOF. Direct computation.

LEMMA 2. *Let T be a bounded operator on H and let $\mathcal{G}_1$ be a connected open set in $\rho_r(T)$ such that dimension*$(\ker(\lambda - T)) \equiv 1$, $\lambda \in \mathcal{G}_1$. *For any $\varepsilon > 0$ there is a right resolvent R of T on $\mathcal{G}_1$ except for an at most countable set S, which does not accumulate in $\mathcal{G}_1$, and satisfies* $\operatorname{dist}(S, \operatorname{bdry} \mathcal{G}_1) < \varepsilon$.

PROOF. There is a vector $y \in H$ such that the orthogonal projection $P_{\ker(\lambda - T)}$ onto the space $\ker(\lambda - T)$, satisfies $P_{\ker(\lambda - T)} y \neq 0$ for all $\lambda \in \mathcal{G}_1$ except for an at most countable set S which does not accumulate in $\mathcal{G}_1$ and such that $\operatorname{dist}(S, \operatorname{bdry} \mathcal{G}_1) < \varepsilon$ (for a proof see [3, Proposition 1.8]). Fix $\lambda_0 \in \mathcal{G}_1 \setminus S$ and let φ be a nonzero vector in $\ker(\lambda_0 - T)$. If R_0 is a fixed right inverse of $\lambda_0 - T$, then any right inverse of $\lambda_0 - T$ is of the form $R_f = R_0 + \langle , f \rangle \varphi$, where $f \in H$. There is a choice of $f_0 \in H$ such that $y \perp \operatorname{Range} R_{f_0}$.

In order to see this observe the orthogonal complement of the range R_f is the null space of $R_0^* + \langle \cdot, \varphi \rangle f$. Now $R_0^* y + \langle y, \varphi \rangle f_0 = 0$, when $f_0 = -\langle y, \varphi \rangle^{-1} R_0^* y$. This last vector is well defined since $P_{\ker(\lambda_0 - T)} y \neq 0$ and, therefore, $\langle y, \varphi \rangle \neq 0$.

For $\lambda \in \mathcal{G}_1$ the following identity holds:

$$(\lambda - T) R_{f_0} = ((\lambda - \lambda_0) R_{f_0} + I). \tag{2}$$

This shows that $\lambda \to -(\lambda - \lambda_0)^{-1}$ is a mapping of $\mathcal{G}_1$ into the component of $\rho_F(R_{f_0})$ which contains the point at infinity. Also, for $\lambda \in \mathcal{G}_1$, the Fredholm index of $(\lambda - \lambda_0) R_{f_0} + I$ is zero. Suppose, for some $\lambda_1 \in \mathcal{G}_1 \setminus S$, that $\ker[(\lambda_1 - \lambda_0) R_{f_0} + I] \neq (0)$. Then from (2) it follows that, for some $x \neq 0$, $(\lambda_1 - T) R_{f_0} x = 0$. In this case, $R_{f_0} x \in \ker(\lambda_1 - T)$ and since this last space is one dimensional, $\ker(\lambda_1 - T) \subset \operatorname{Range} R_{f_0}$. This contradicts $y \perp \operatorname{Range} R_{f_0}$ and $P_{\ker(\lambda_1 - T)} y \neq 0$. It is now clear that, for λ in $\mathcal{G}_1 \setminus S$, the operator $(\lambda - \lambda_0) R_{f_0} + I$ is invertible. The operator valued function $R(\lambda) = R_{f_0}((\lambda - \lambda_0) R_{f_0} + I)^{-1}$ is a right resolvent for T in $\mathcal{G}_1 \setminus S$. This completes the proof.

LEMMA 3. *Let T be a bounded operator on H and let $\mathcal{G}_n$ be an open connected subset in $\rho_r(T)$. Assume that* dimension$(\ker(\lambda - T)) = n$, *for $\lambda \in \mathcal{G}_n$. Then for any $\varepsilon > 0$, there is a right resolvent for T on $\mathcal{G}_n$ except for an at most countable set $S \subset \mathcal{G}_n$ which does not accumulate in $\mathcal{G}_n$, such that* $\operatorname{dist}(S, \operatorname{bdry} \mathcal{G}_n) < \varepsilon$.

PROOF. The proof proceeds by induction on n. The result is clear if $n = 0$

for then $\mathcal{G}_n$ is a subset of the resolvent set of T. The case $n = 1$ is contained in the preceding lemma. Suppose the result has been obtained in case $n = k - 1$. Let $\mathcal{G}_k$ be a connected open subset of $\rho_r(T)$ such that dimension $\ker(\lambda - T) = k$, $\lambda \in \mathcal{G}_k$. For any $\varepsilon > 0$ there is a vector $y \in H$ for which $P_{\ker(\lambda - T)} y \neq 0$, for all $\lambda \in \mathcal{G}_k \setminus S'$; where S' is an at most countable set which does not accumulate in $\mathcal{G}_k$ and satisfies $\operatorname{dist}(S', \operatorname{bdry} \mathcal{G}_k) < \varepsilon$ (see [3, Proposition 1.8]). Let $Y_\lambda = \ker(\lambda - T) \cap \{y\}^\perp$, for $\lambda \in \mathcal{G}_k$, and let $Y = \text{c.l.m.}_{\lambda \in \mathcal{G}_k}\{Y_\lambda\}$; here, c.l.m. is an abbreviation for closed linear manifold. Obviously, $TY \subset Y$ and relative to the decomposition $H = Y \oplus Y^\perp$,

$$T = \begin{pmatrix} T_Y & * \\ 0 & T_{Y^\perp} \end{pmatrix};$$

here, T_Y is the restriction of T to Y and $T_{Y^\perp}$ is the compression of T to $Y^\perp$. It is easy to establish that $\lambda - T_Y$ is onto for $\lambda \in \mathcal{G}_k$ and clearly $(\lambda - T_{Y^\perp})$ is onto for $\lambda \in \mathcal{G}_k$. It follows that for $\lambda \in \mathcal{G}_k \setminus S'$, dimension $\ker(\lambda - T_Y) = k - 1$ and dimension $\ker(\lambda - T_{Y^\perp}) = 1$. The induction hypothesis can be combined with Lemma 1 and Lemma 2 to complete the proof.

Following [3] we introduce the notations

$H_r(T) = \text{c.l.m.}_{\lambda \in \rho^r_{s-F}(T)} \ker(\lambda - T)$,
$H_l(T) = \text{c.l.m.}_{\lambda \in \rho^r_{s-F}(T)} \ker(\lambda - T)^*$,
$H_0(T) = H \ominus (H_r(T) \oplus H_l(T))$.

The proof that $H_r(T)$ and $H_l(T)$ are orthogonal subspaces appears in [3].

Relative to the decomposition $H = H_r(T) \oplus H_0(T) \oplus H_l(T)$ the operator T has the 3×3 matrix form

$$T = \begin{bmatrix} T_r & A & B \\ 0 & T_0 & C \\ 0 & 0 & T_l \end{bmatrix}. \tag{3}$$

The relevant spectral properties of T_r, T_0, T_l are:

(i) $\rho_{s-F}(T) \subset \rho_r(T_r) \cap \rho_l(T_l)$,

(ii) $\rho^r_{s-F}(T) \subset \rho(T_0)$ (the resolvent set of T_0),

(iii) $\rho^s_{s-F}(T)$ is a subset of the isolated eigenvalues of T_0 which have finite algebraic multiplicity.

The proofs of the inclusions (i)–(iii) appear in [3].

Next let $\mathcal{G}$ be an open set and T an operator having the 3×3 matrix form (3) relative to the decomposition $H = H_r(T) \oplus H_0(T) \oplus H_l(T)$. Assume further that T_r has a right inverse function R in $\mathcal{G}$, T_l has a left inverse function L in $\mathcal{G}$ and that $\mathcal{G} \subset \rho(T_0)$.

For λ in $\mathcal{G}$, set

$$F(\lambda) = \begin{bmatrix} R(\lambda) & R(\lambda)AR(\lambda\colon T_0) & R(\lambda)[AR(\lambda\colon T_0)C + B]L(\lambda) \\ 0 & R(\lambda\colon T_0) & R(\lambda\colon T_0)CL(\lambda) \\ 0 & 0 & L(\lambda) \end{bmatrix}; \tag{4}$$

here, $R(\lambda\colon T_0) = (\lambda - T_0)^{-1}$.

The operator valued function has the following properties [4]:

(a) F is a generalized inverse function for F in $\mathcal{G}$.

(b) If R, L are analytic in $\mathcal{G}$, then F is an analytic generalized inverse function for T in $\mathcal{G}$.

(c) If R, L are right and left resolvents of T_r, T_l, respectively, then F is a generalized resolvent of T in $\mathcal{G}$.

2. **Proof of Theorem 1.** Let $\varepsilon > 0$. It suffices to construct a generalized resolvent for T on each component $\mathcal{G}$ of $\rho_F^r(T)$ except for an at most countable set S which does not accumulate in $\mathcal{G}$ and satisfies $\operatorname{dist}(S, \operatorname{bdry} \mathcal{G}) < \varepsilon$. For such a $\mathcal{G}$ we have $\mathcal{G} \subset \rho_r(T_r) \cap \rho_l(T_l)$. Moreover,

$$\text{dimension} \ker(\lambda - T_r) \equiv n < \infty,$$

$$\text{dimension} \ker(\lambda - T_l)^* \equiv m < \infty \quad \text{on } \mathcal{G}.$$

It follows from Lemma 3 that we can construct a right resolvent R for T_r in all of $\mathcal{G}$ except for an at most countable set S' which does not accumulate in $\mathcal{G}$ and satisfies $\operatorname{dist}(S', \operatorname{bdry} \mathcal{G}) < \varepsilon$. The same argument applied to T_l^* in $\mathcal{G}^*$ implies the existence of a left resolvent L for T_l in all of $\mathcal{G}$ except for an at most countable set S'' which does not accumulate in $\mathcal{G}$ and satisfies $\operatorname{dist}(S'', \operatorname{bdry} \mathcal{G}) < \varepsilon$.

From the inclusion $\mathcal{G} \subset \rho(T_0)$ it follows that the function F defined by (4) is a generalized resolvent for T in $\mathcal{G}$ except for the at most countable set $S = S' \cup S''$, where S does not accumulate in $\mathcal{G}$ and satisfies $\operatorname{dist}(S, \operatorname{bdry} \mathcal{G}) < \varepsilon$. This completes the proof.

REMARK. It would be possible to extend the above argument to construct a generalized resolvent in all of $\rho_r(T)$ if the following question has an affirmative answer.

Question. Let $\mathcal{G}$ be an open connected subset in $\rho_r(T)$. Assume

$$\text{dimension} \ker(\lambda - T) = 1, \quad \text{for all } \lambda \in \mathcal{G}.$$

Does there exist a y in H such that $P_{\ker(\lambda - T)} y \neq 0$, for all $\lambda \in \mathcal{G}$?

In connection with the above question we refer the reader to [8] and the example of A. Douady. In [8, p. 244] an example is given of a holomorphic Hilbert space valued function on the unit disc D with the following properties: (i) For each compact subset $K \subset D$, there is an $x \in H$ such that $(h(z), x)$ has no zeroes on K. (ii) For all $y \in H$, the function $(h(z), y)$ has a zero in D.

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