ON GENERALIZED RESOLVENTS

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Abstract. Let $T$ be a bounded linear operator on a Hilbert space and $\rho_f(T)$ the Fredholm domain of $T$. It is shown that a generalized resolvent can be constructed for $T$ in $\rho_F(T)$ which verifies the resolvent identity except for an at most countable set of points which are close to the boundary of $\rho_F(T)$.

Let $T$ be a bounded linear operator on a Hilbert space $H$. In case the range of $T$ is a closed subspace of $H$, then an operator $F$ will be called a generalized inverse of $T$ when $FT$ is a projection onto the orthogonal complement of the kernel of $T$ and $TF$ is a projection onto the range of $T$. Unless $T$ is invertible, then a generalized inverse is not unique. Let $\mathcal{G}$ be a domain in the complex plane $\mathbb{C}$ such that for every $\lambda$ in $\mathcal{G}$, the operator $\lambda - T$ has closed range. An operator valued function $F$ defined on $\mathcal{G}$ is called a generalized inverse function for $T$ on an open set $\mathcal{G}$, when for every pair $\lambda$, $\mu$ in a component of $\mathcal{G}$

$$F(\lambda) - F(\mu) = (\mu - \lambda)F(\lambda)F(\mu).$$

A continuous generalized inverse function, for an operator $T$ on an open set $\mathcal{G}$, which verifies the resolvent identity on $\mathcal{G}$, will be called a generalized resolvent on $\mathcal{G}$.

This note is concerned with the construction of generalized resolvents on open subsets of the Fredholm domain of a bounded operator $T$. Recall that an operator $T$ is called semi-Fredholm in case $T$ has closed range and the dimension of at least one of ker($T$) or ker($T^*$) is finite; here, ker denotes kernel and $T^*$ is the adjoint of $T$. If $T$ has closed range and both ker($T$) and ker($T^*$) are finite dimensional, then $T$ is called a Fredholm operator. The semi-Fredholm domain of $T$ is the set $\rho_{s-F}(T) = \{\lambda \in \mathbb{C}: \lambda - T \text{ is semi-Fredholm}\}$ and the Fredholm domain of $T$ is the set $\rho_F(T) = \{\lambda \in \mathbb{C}: \lambda - T \text{ is Fredholm}\}$.

There is one obvious obstruction to constructing a generalized resolvent for $T$ on all of $\rho_{s-F}(T)$. In $\rho_{s-F}(T)$ there is an at most countable set where the function

$$m(\lambda) = \text{minimum dimension} [\text{ker}(\lambda - T), \text{ker}(\lambda - T)^*]$$

is discontinuous [3, Proposition 2.6], [5], [6]. This set will be denoted by $\rho_{s-F}(T)$ and is referred to as the set of singular points in the semi-Fredholm
The singular points in the Fredholm domain $\rho_{s-F}(T) \cap \rho_F(T)$ will be denoted by $\rho_F^s(T)$. The set $\rho_{s-F}(T)$ does not accumulate in $\rho_{s-F}(T)$ (see, e.g., [3], [5], [6]) and it is easy to show $\rho_F(T) \cap \rho_{s-F}(T) = \emptyset$. The complementary set $\rho_{s-F}(T) = \rho_{s-F}(T) \setminus \rho_{s-F}(T)$ ($\rho_F^s(T) = \rho_F(T) \setminus \rho_F^s(T)$) is called the set of regular points in the semi-Fredholm (Fredholm) domain. Obviously, there does not exist a continuous generalized inverse function in a neighborhood of a point $\lambda \in \rho_{s-F}(T)$.

The notation $\text{bdry} \ S$ will be used for the boundary of a subset of $\mathbb{C}$ and $\text{dist}(S, S')$ will denote the Hausdorff distance between two bounded sets $S, S'$ in $\mathbb{C}$. In other words,

$$\text{dist}(S, S') = \max \left[ \sup_{\lambda \in S'} [\text{distance}(\lambda, S)], \sup_{\lambda \in S} [\text{distance}(\lambda, S')] \right].$$

The main result to be established here is

**Theorem 1.** Let $T$ be a bounded operator on $H$ and let $\varepsilon > 0$. There exists a generalized resolvent on $\rho_{s-F}(T)$ except for an at most countable set $S$ which does not accumulate in $\rho_F(T)$. Moreover, $\text{dist}(S, \text{bdry} \rho_F(T)) < \varepsilon$.

There are several papers in the literature which contain results similar in spirit to the above theorem. In [8] P. Saphar obtains the above theorem (in the generality of operators on a Banach space) with the conclusion

"$\text{dist}(S, \text{bdry} \rho_F(T)) < \varepsilon$"

replaced by

"$\text{dist}(S, \text{bdry} \rho_F^s(T)) < \varepsilon$".

Also Shapiro and Schechter [9] construct generalized resolvents on $\rho_F^s(T)$, minus a countable set $S$ for operators $T$ acting on a Banach space. These authors do not make any attempt to push the set $S$ out near the boundary of $\rho_F(T)$.

It is clear that a generalized resolvent for $T$ defined on an open set $\mathcal{O}$ is an analytic generalized inverse function for $T$ in $\mathcal{O}$. On the other hand, not every analytic generalized inverse function defined on an open set $\mathcal{O}$ verifies the resolvent identity on $\mathcal{O}$. Let $\rho_T(T)$ $(\rho_T(T))$ denote the set of complex $\lambda$, where $\lambda - T$ has a right (left) inverse. Allan [1], [2] has shown that there exists an analytic right (left) inverse function for $T$ in $\rho_T(T)$ ($\rho_T(T)$). This fact can be used to construct an analytic generalized inverse function for an operator $T$ in $\rho_{s-F}(T)$ [4].

Using the result in Theorem 1 it is possible to construct a "generalized spectral projection" associated with any finite subset $\sigma$ of $\rho_F^s(T)$. This leads to a decomposition of the operator $T$ as a direct sum $T_1 \oplus T_2$ (not necessarily an orthogonal sum), where $T_1$ contains the singular points $\sigma$ as isolated eigenvalues in its spectrum and the operator $T_2$ satisfies $\rho_F(T_2) = \rho_F^s(T) \cup \sigma$. The reader is referred to [4] for further details.

1. Preliminaries. This section begins with a few basic lemmas:

**Lemma 1.** Let $T$ be an operator on $H$ and let $H_1, H_2$ be closed subspaces of $H$. Assume that relative to the orthogonal decomposition $H = H_1 \oplus H_2$ the
operator $T$ has the $2 \times 2$ matrix representation

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Let $\mathcal{G}$ be an open set on which both $A$, $B$ have right resolvents $R_1$, $R_2$ respectively. Then

$$R = \begin{pmatrix} R_1 & R_1CR_2 \\ 0 & R_2 \end{pmatrix}$$

defines a right resolvent of $T$ on $\mathcal{G}$.

**Proof.** Direct computation.

**Lemma 2.** Let $T$ be a bounded operator on $H$ and let $\mathcal{G}_1$ be a connected open set in $\rho_p(T)$ such that $\text{dimension}(\ker(\lambda - T)) = 1$, $\lambda \in \mathcal{G}_1$. For any $\epsilon > 0$ there is a right resolvent $R$ of $T$ on $\mathcal{G}_1$ except for an at most countable set $S$, which does not accumulate in $\mathcal{G}_1$, and satisfies $\text{dist}(S, \text{bdry } \mathcal{G}_1) < \epsilon$.

**Proof.** There is a vector $y \in H$ such that the orthogonal projection $P_{\ker(\lambda - T)}$ onto the space $\ker(\lambda - T)$, satisfies $P_{\ker(\lambda - T)}y \neq 0$ for all $\lambda \in \mathcal{G}_1$ except for an at most countable set $S$ which does not accumulate in $\mathcal{G}_1$ and such that $\text{dist}(S, \text{bdry } \mathcal{G}_1) < \epsilon$ (for a proof see [3, Proposition 1.8]). Fix $\lambda_0 \in \mathcal{G}_1 \setminus S$ and let $\varphi$ be a nonzero vector in $\ker(\lambda_0 - T)$. If $R_0$ is a fixed right inverse of $\lambda_0 - T$, then any right inverse of $\lambda_0 - T$ is of the form $R_j = R_0 + \langle \cdot, f \rangle \varphi$, where $f \in H$. There is a choice of $f_0 \in H$ such that $y \perp \text{Range } R_{f_0}$.

In order to see this observe the orthogonal complement of the range $R_f$ is the null space of $R_f^* + \langle \cdot, \varphi \rangle f$. Now $R_{f_0}^* y + \langle y, \varphi \rangle f_0 = 0$, when $f_0 = -\langle y, \varphi \rangle^{-1} R_{f_0} y$. This last vector is well defined since $P_{\ker(\lambda_0 - T)} y \neq 0$ and, therefore, $\langle y, \varphi \rangle \neq 0$.

For $\lambda \in \mathcal{G}_1$ the following identity holds:

$$\lambda - T)R_{f_0} = ((\lambda - \lambda_0)R_{f_0} + I).$$

This shows that $\lambda \mapsto (\lambda - \lambda_0)^{-1}$ is a mapping of $\mathcal{G}_1$ into the component of $\rho_F(R_{f_0})$ which contains the point at infinity. Also, for $\lambda \in \mathcal{G}_1$, the Fredholm index of $(\lambda - \lambda_0)R_{f_0} + I$ is zero. Suppose, for some $\lambda_1 \in \mathcal{G}_1 \setminus S$, that $\ker((\lambda_1 - \lambda_0)R_{f_0} + I) \neq (0)$. Then from (2) it follows that, for some $x \neq 0$, $(\lambda_1 - T)R_{f_0} x = 0$. In this case, $R_{f_0} x \in \ker(\lambda_1 - T)$ and since this last space is one dimensional, $\ker(\lambda_1 - T) \subset \text{Range } R_{f_0}$. This contradicts $y \perp \text{Range } R_{f_0}$ and $P_{\ker(\lambda_1 - T)} y \neq 0$. It is now clear that, for $\lambda$ in $\mathcal{G}_1 \setminus S$, the operator $(\lambda - \lambda_0)R_{f_0} + I$ is invertible. The operator valued function $R(\lambda) = R_{f_0}((\lambda - \lambda_0)R_{f_0} + I)^{-1}$ is a right resolvent for $T$ in $\mathcal{G}_1 \setminus S$. This completes the proof.

**Lemma 3.** Let $T$ be a bounded operator on $H$ and let $\mathcal{G}_n$ be an open connected subset in $\rho_p(T)$. Assume that dimension($\ker(\lambda - T)$) = $n$, for $\lambda \in \mathcal{G}_n$. Then for any $\epsilon > 0$, there is a right resolvent for $T$ on $\mathcal{G}_n$ except for an at most countable set $S \subset \mathcal{G}_n$ which does not accumulate in $\mathcal{G}_n$, such that $\text{dist}(S, \text{bdry } \mathcal{G}_n) < \epsilon$.

**Proof.** The proof proceeds by induction on $n$. The result is clear if $n = 0$.
for then \( \mathcal{G}_n \) is a subset of the resolvent set of \( T \). The case \( n = 1 \) is contained in the preceding lemma. Suppose the result has been obtained in case \( n = k - 1 \). Let \( \mathcal{G}_k \) be a connected open subset of \( \rho_r(T) \) such that dimension \( \ker(\lambda - T) = k, \lambda \in \mathcal{G}_k \). For any \( \varepsilon > 0 \) there is a vector \( y \in H \) for which \( P_{\ker(\lambda - T),y} \neq 0 \), for all \( \lambda \in \mathcal{G}_k \setminus S' \); where \( S' \) is an at most countable set which does not accumulate in \( \mathcal{G}_k \) and satisfies \( \text{dist}(S', \text{bdry } \mathcal{G}_k) < \varepsilon \) (see [3, Proposition 1.8]). Let \( Y_\lambda = \ker(\lambda - T) \cap \{y\}^\perp \), for \( \lambda \in \mathcal{G}_k \), and let \( Y = \text{c.l.m.}_\lambda \in \mathcal{G}_k \{Y_\lambda\} \); here, c.l.m. is an abbreviation for closed linear manifold. Obviously, \( TY \subset Y \) and relative to the decomposition \( H = Y \oplus Y^\perp \),

\[
T = \begin{pmatrix}
T_Y & * \\
0 & T_{Y^\perp}
\end{pmatrix};
\]

here, \( T_Y \) is the restriction of \( T \) to \( Y \) and \( T_{Y^\perp} \) is the compression of \( T \) to \( Y^\perp \). It is easy to establish that \( \lambda - T_Y \) is onto for \( \lambda \in \mathcal{G}_k \) and clearly \( (\lambda - T_{Y^\perp}) \) is onto for \( \lambda \in \mathcal{G}_k \). It follows that for \( \lambda \in \mathcal{G}_k \setminus S' \), dimension \( \ker(\lambda - T_Y) = k - 1 \) and dimension \( \ker(\lambda - T_{Y^\perp}) = 1 \). The induction hypothesis can be combined with Lemma 1 and Lemma 2 to complete the proof.

Following [3] we introduce the notations

\[
\begin{align*}
H_r(T) &= \text{c.l.m.}_\lambda \in \rho_r(T) \ker(\lambda - T), \\
H_i(T) &= \text{c.l.m.}_\lambda \in \rho_i(T) \ker(\lambda - T)^*, \\
H_0(T) &= H \ominus (H_r(T) \oplus H_i(T)).
\end{align*}
\]

The proof that \( H_r(T) \) and \( H_i(T) \) are orthogonal subspaces appears in [3].

Relative to the decomposition \( H = H_r(T) \oplus H_0(T) \oplus H_i(T) \) the operator \( T \) has the \( 3 \times 3 \) matrix form

\[
(3) \quad T = \begin{pmatrix}
T_r & A & B \\
0 & T_0 & C \\
0 & 0 & T_i
\end{pmatrix}.
\]

The relevant spectral properties of \( T_r, T_0, T_i \) are:

(i) \( \rho_r(T) \subset \rho_r(T_r) \cap \rho_i(T_i) \),

(ii) \( \rho_r(T) \subset \rho(T_0) \) (the resolvent set of \( T_0 \)),

(iii) \( \rho_i(T) \) is a subset of the isolated eigenvalues of \( T_0 \) which have finite algebraic multiplicity.

The proofs of the inclusions (i)–(iii) appear in [3].

Next let \( \mathcal{G} \) be an open set and \( T \) an operator having the \( 3 \times 3 \) matrix form (3) relative to the decomposition \( H = H_r(T) \oplus H_0(T) \oplus H_i(T) \). Assume further that \( T_r \) has a right inverse function \( R \) in \( \mathcal{G} \), \( T_i \) has a left inverse function \( L \) in \( \mathcal{G} \) and that \( \mathcal{G} \subset \rho(T_0) \).

For \( \lambda \in \mathcal{G} \), set

\[
(4) \quad F(\lambda) = \begin{pmatrix}
R(\lambda) & R(\lambda)AR(\lambda; T_0) & R(\lambda)[AR(\lambda; T_0)C + B]L(\lambda) \\
0 & R(\lambda; T_0) & R(\lambda; T_0)CL(\lambda) \\
0 & 0 & L(\lambda)
\end{pmatrix};
\]

here, \( R(\lambda; T_0) = (\lambda - T_0)^{-1} \).

The operator valued function has the following properties [4]:

(a) \( F \) is a generalized inverse function for \( F \) in \( \mathcal{G} \).
(b) If $R, L$ are analytic in $\mathfrak{S}$, then $F$ is an analytic generalized inverse function for $T$ in $\mathfrak{S}$.

(c) If $R, L$ are right and left resolvents of $T_r, T_l$, respectively, then $F$ is a generalized resolvent of $T$ in $\mathfrak{S}$.

2. Proof of Theorem 1. Let $\varepsilon > 0$. It suffices to construct a generalized resolvent for $T$ on each component $\mathfrak{S}$ of $\rho F(T)$ except for an at most countable set $S$ which does not accumulate in $\mathfrak{S}$ and satisfies $\text{dist}(S, \text{bdry } \mathfrak{S}) < \varepsilon$. For such a $\mathfrak{S}$ we have $\mathfrak{S} \subset \rho_r(T_r) \cap \rho_l(T_l)$. Moreover,

$$\text{dimension ker}(\lambda - T_r) \equiv n < \infty,$$
$$\text{dimension ker}(\lambda - T_l)^* \equiv m < \infty \quad \text{on } \mathfrak{S}.$$

It follows from Lemma 3 that we can construct a right resolvent $R$ for $T_r$ in all of $\mathfrak{S}$ except for an at most countable set $S'$ which does not accumulate in $\mathfrak{S}$ and satisfies $\text{dist}(S', \text{bdry } \mathfrak{S}) < \varepsilon$. The same argument applied to $T_l^*$ in $\mathfrak{S}^*$ implies the existence of a left resolvent $L$ for $T_l$ in all of $\mathfrak{S}$ except for an at most countable set $S''$ which does not accumulate in $\mathfrak{S}$ and satisfies $\text{dist}(S'', \text{bdry } \mathfrak{S}) < \varepsilon$.

From the inclusion $\mathfrak{S} \subset \rho(T_0)$ it follows that the function $F$ defined by (4) is a generalized resolvent for $T$ in $\mathfrak{S}$ except for the at most countable set $S = S' \cup S''$, where $S$ does not accumulate in $\mathfrak{S}$ and satisfies $\text{dist}(S, \text{bdry } \mathfrak{S}) < \varepsilon$. This completes the proof.

Remark. It would be possible to extend the above argument to construct a generalized resolvent in all of $\rho_r(T)$ if the following question has an affirmative answer.

**Question.** Let $\mathfrak{S}$ be an open connected subset in $\rho_r(T)$. Assume

$$\text{dimension ker}(\lambda - T) = 1, \quad \text{for all } \lambda \in \mathfrak{S}.$$

Does there exist a $y$ in $H$ such that $P_{\ker(\lambda - T)} y \neq 0$, for all $\lambda \in \mathfrak{S}$?

In connection with the above question we refer the reader to [8] and the example of A. Douady. In [8, p. 244] an example is given of a holomorphic Hilbert space valued function on the unit disc $D$ with the following properties: (i) For each compact subset $K \subset D$, there is an $x \in H$ such that $(h(z), x)$ has no zeroes on $K$. (ii) For all $y \in H$, the function $(h(z), y)$ has a zero in $D$.

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References


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