AN \textit{A}-PROPER MAP WITH PRESCRIBED
TOPOLOGICAL DEGREE

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\textbf{Abstract.} For any given element $a$ of the ring $\mathcal{Z} = \mathbb{Z}_N/I$, where $I$ is the ideal of integer sequences convergent to 0, an \textit{A}-proper map in $l_2$ is constructed whose degree in the sense of S. F. Wong is equal to $a$.

The concept of an \textit{A}-proper map acting in a Banach space was introduced by Petryshyn [6, p. 157] (originally called an operator satisfying condition (H)). A topological degree for such maps was defined by Browder and Petryshyn [1], [2] as a generalization of the Leray-Schauder degree [5] for maps of the form \textit{Identity} + Compact. Browder and Petryshyn established the basic properties of their degree, $\text{Deg} (T, G, y)$, which is set valued (see the note after Definition 3 for its definition), invariant under suitable homotopies, satisfies a weakened sum formula

$$\text{Deg} (T, G, y) \subseteq \text{Deg} (T, G_1, y) + \text{Deg} (T, G_2, y),$$

and if $\text{Deg} (T, G, y) \neq \{0\}$ then there is an $x \in G$ such that $T(x) = y$. Wong [9] has given a new definition of the degree with values in a ring $\mathcal{Z}$ (see below) which satisfies the sum formula with an equality sign. Later Wong [10] proved a restricted product formula for the degree of the product $TU$ under the restriction that at least one of the maps $T$ or $U$ must be of the form \textit{Identity} + Compact.

The purpose of this paper is, given $a \in \mathcal{Z}$, to construct an \textit{A}-proper map in $l_2$ whose degree is $a$. In the following let $X$ be a real Banach space. Let $\text{cl} (G)$ denote the closure of $G$ and $\partial G$ the topological boundary of $G$ for subsets $G$ of $X$.

**Definition 1.** An (oriented) projectionally complete scheme $\Gamma$ for mappings from subsets of $X$ to $X$ is a monotonically increasing sequence $\{X_n\}$ of (oriented) finite dimensional subspaces of $X$ and a sequence $\{P_n\}$ of continuous linear projections $P_n : X \to X_n$ with $P_n X = X_n$, such that $P_n x \to x$ as $n \to \infty$ for each $x \in X$.

This definition is adapted from Fitzpatrick [4, Definition 1.1, p. 537]. The following definition is that of Petryshyn [7, Definition 1, p. 271].

**Definition 2.** Let $G$ be a subset of $X$, and $\Gamma$ a projectionally complete

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scheme in the sense of Definition 1. The mapping $T : \text{cl} (G) \to X$ is $A$-proper with respect to $\Gamma$ if for any bounded sequence $\{x_n\}$ with $x_n \in \text{cl} (G) \cap X_n$ such that $P_n T(x_n) \to g \in X$, there exists a subsequence $\{x_{n(\ell)}\}$ and an $x \in \text{cl} (G)$ such that $x_{n(\ell)} \to x$ as $\ell \to \infty$ and $T(x) = g$.

Such mappings include mappings of the form $I + C$ where $I$ is the identity and $C$ is compact [6, Remark 3, p. 162], mappings of the form $I + S + C$ under certain conditions where $S$ is strictly contractive [6, Theorem 7, p. 162], and $K$-monotone mappings under certain conditions [7, Corollary 2.1, p. 220 and Theorem 2.3, p. 222]. This latter class includes monotone mappings [7, p. 228] and $J$-monotone or accretive mappings [7, pp. 230–231].

The following definition is adapted from Wong [9, p. 406] and makes use of the classical degree in $R^n$, $\text{deg} (f, D, q)$, called the Brouwer degree of $f$ at $q$ relative to $D$ (cf. [3, Definition 6.3, p. 31] or [8, Definition 3.14, p. 71]). Here $D$ is a bounded open set in oriented Euclidean $n$-space $R^n$, $f$ is a continuous mapping from $\text{cl} (D)$ into $R^n$, and $q \in f(\partial D)$. By $\mathcal{Z}$ we denote the ring of all equivalence classes $[s_n] = \{(t_n) : t_n = s_n$ for all $n$ sufficiently large$\}$ of sequences of integers.

**Definition 3.** Let $T : (G) \to X$ be $A$-proper with respect to a given approximation scheme, where $G_n = G \cap X_n$ is bounded and open in $X_n$ for all $n$ sufficiently large and $T_n = P_n T|_{G_n}$ is continuous for all $n$ sufficiently large. Let $y \in X \setminus T(\partial G)$. Then the degree of $T$ at $y$ relative to $G$ is the element $D(T, G, y) = [s_n]$ of $\mathcal{Z}$ such that

$$s_n = \text{deg} (T_n, G_n, P_n y)$$

for all $n$ sufficiently large.

Note that $\text{deg} (T_n, G_n, P_n y)$ is defined for all $n$ sufficiently large since $P_n y \notin T_n(\partial G_n)$ for all $n$ sufficiently large by [2, Lemma 1, p. 220]. The degree of Browder and Petryshyn [1], $\text{Deg} (T, G, y)$, is the set of limit points of $\{\text{deg} (T_n, G_n, P_n y)\}$ including possibly $\pm \infty$.

In the following the Banach space $l_2$ of square summable real sequences with norm $\| (\alpha_i) \|^2 = \sum_{i=1}^{\infty} \alpha_i^2$ will have the oriented projectionally complete scheme $\Gamma(l_2)$ given by

$$X_n = \text{span} \{e_1, e_2, \ldots, e_n\} \quad \text{for } n = 1, 2, \ldots$$

(where $e_i$ has coordinate 1 in the $i$th place and 0 elsewhere) and

$$P_n \left( \sum_{i=1}^{\infty} \alpha_i e_i \right) = \sum_{i=1}^{n} \alpha_i e_i \quad \text{for } n = 1, 2, \ldots$$

The orientation of $X_n$ is determined by the order $(e_1, e_2, \ldots, e_n)$ of the basis elements. Let $H_m$ be the subset of $l_2$ given by

$$H_m = \{x \in l_2 : \|m e_m - x\| < \frac{1}{2}\}$$

and let $G$ be given by

$$G = \bigcup_{m=1}^{\infty} H_m.$$
Then, as is shown in Appendix I, the $H_m$ have disjoint closures, $\text{cl} (H_m) \cap X_n$ is empty for all $m > n$, and $\text{cl} (G) = \bigcup_{m=1}^{\infty} \text{cl} (H_m)$.

**THEOREM.** Given any element $[s_n]$ of $\mathbb{Z}$, there is a mapping $T: \text{cl} (G) \to l_2$ which is $A$-proper with respect to $\Gamma(l_2)$ and such that $D(T, G, 0) = [s_n]$.

**PROOF.** Let $t_0 = t_1 = 0$ and $t_n = s_n$ if $n \geq 2$. Then put

$$k_n = |t_n - t_{n-1}|, \quad \epsilon_n = \text{sign}(t_n - t_{n-1}),$$

and

$$a_n = 2n - 1, \quad b_n = 2n, \quad n = 1, 2, \ldots.$$

Define the mapping $T$: $\text{cl} (G) \to l_2$ as follows. For $x = \sum_{i=1}^{\infty} \alpha_i e_i \in \text{cl} (G)$ there is a unique $m$ such that $x \in \text{cl} (H_m)$. Then put $T(x) = \sum_{i=1}^{\infty} \eta_i e_i$ where

$$\eta_{m-1} = \epsilon_m \prod_{i=1}^{k_m} (\alpha_m - m - a_i \alpha_{m-1}),$$

$$\eta_m = \prod_{i=1}^{k_m} (\alpha_m - m - b_i \alpha_{m-1}),$$

$$\eta_i = \alpha_i \quad \text{for all } i \neq m - 1, m,$$

for $m \geq 2$, and $\eta_i = \alpha_i$ for all $i$ if $m = 1$.

First we show $T$ is $A$-proper. Let $\{x_n \in G_n\}$ be a bounded sequence and $g \in l_2$ be such that $P_n^q T(x_n) \to g$. Let

$$x_n = \sum_{i=1}^{n} \alpha_{i,n} e_i, \quad P_n^q T(x_n) = \sum_{i=1}^{n} \beta_{i,n} e_i, \quad \text{and} \quad g = \sum_{i=1}^{\infty} \gamma_i e_i.$$

Since $\{x_n\}$ is bounded, there is a $p$ such that $\{x_n\} \subseteq \bigcup_{m=1}^{p} \text{cl} (H_m)$, for if $y \in \text{cl} (H_m)$ then $||y|| > ||me_m|| - ||y - me_m|| > m - \frac{1}{2}$. There is a subsequence of $\{x_n\}$ (call it $\{x_{n_j}\}$) again) and a $q \in \{1, \ldots, p\}$ such that $\{x_{n_j}\} \subseteq \text{cl} (H_q)$.

Now $\{\alpha_{i,n_j}\}_{j=1}^{\infty}$ is bounded for each fixed $i$, since $|\alpha_{i,n_j}| \leq \frac{1}{2}$ for $i \neq q$ and $|\alpha_{q,n_j}| \leq q + \frac{1}{2}$. Hence there is a further subsequence $\{x_{n_j(k)}\}$ of $\{x_{n_j}\}$ and $\alpha_i (i = 1, \ldots, q)$ such that $\alpha_{i,n_j(k)} \to \alpha_i$ as $k \to \infty$, $i = 1, \ldots, q$. Put $\alpha_i = \gamma_i$ for $i > q$, and $x = \sum_{i=1}^{\infty} \alpha_i e_i$. Then

$$||x_{n_j(k)} - x|| \leq \sum_{i=1}^{q} ||\alpha_{i,n_j(k)} e_i - \alpha_i e_i|| + \sum_{i=q+1}^{\infty} ||\alpha_{i,n_j(k)} e_i - \sum_{i=q+1}^{\infty} \alpha_i e_i||$$

$$= \sum_{i=1}^{q} ||\alpha_{i,n_j(k)} - \alpha_i|| + \sum_{i=q+1}^{\infty} ||\beta_{i,n_j(k)} - \gamma_i|| e_i||$$

$$\leq \sum_{i=1}^{q} ||\alpha_{i,n_j(k)} - \alpha_i|| + ||P_{n_j(k)} T(x_{n_j(k)}) - g||$$

which tends to zero as $k \to \infty$. Hence $x_{n_j(k)} \to x$ and $x \in l_2$. Indeed $x \in \text{cl} (H_q)$ since $\{x_{n_j}\} \subseteq \text{cl} (H_q)$.

It remains to show that $T(x) = g$. Let $T(x) = \sum_{i=1}^{\infty} \eta_i e_i$. Assume $q > 1$. 

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Since $x \in \text{cl} (H_q)$, $\eta_i = \alpha_i = \gamma_i, i \neq q - 1, q, $

$$\eta_{q-1} = e_q \prod_{i=1}^{k_q} (\alpha_q - q - a_i \alpha_{q-1}) = \lim_{k \to \infty} e_q \prod_{i=1}^{k_q} (\alpha_q \eta_{j(k)} - q - a_i \alpha_{q-1} \eta_{j(k)})$$

$$= \lim_{k \to \infty} \beta_{q-1, \eta_{j(k)}} = \gamma_{q-1},$$

and

$$\eta_q = \prod_{i=1}^{k_q} (\alpha_q - q - b_i \alpha_{q-1}) = \lim_{k \to \infty} \prod_{i=1}^{k_q} (\alpha_q \eta_{j(k)} - q - b_i \alpha_{q-1} \eta_{j(k)})$$

$$= \lim_{k \to \infty} \beta_{q, \eta_{j(k)}} = \gamma_q.$$ 

For the case $q = 1$, since $T$ restricted to $\text{cl} (H_1)$ is the identity, $\eta_i = \alpha_i = \gamma_i$ for all $i = 1, 2, \ldots$. Hence in either case $T(x) = \sum_{i=1}^{\infty} \eta_i e_i = \sum_{i=1}^{\infty} \gamma_i e_i = g$, and $T$ is $A$-proper with respect to $\Gamma (l_2)$.

We will now compute the degree of $T$ at 0 relative to $G$. Since $H_m \cap X_n = \emptyset$ for $m > n$ (see Appendix I), it follows that

$$G_n = G \cap X_n = \bigcup_{m=1}^{n} H_{m,n}$$

where $H_{m,n} = H_m \cap X_n$.

Then

$$\deg (T_n, G_n, 0) = \sum_{m=1}^{n} \deg (T_n, H_{m,n}, 0)$$

by the sum formula for the Brouwer degree (see [3, Theorem 6.8, p. 32] or [8, Theorem 3.16.5, p. 72]) since the $H_m$, and thus the $H_{m,n}$, are disjoint for fixed $n$. Now, for $n \geq m \geq 2$, $T_m$ is the identity on all components except the $(m - 1)$st and $m$th. Thus, by the reduction formula for the Brouwer degree (see [3, Theorem 10.1, p. 51] or [8, Theorem 3.16.7, p. 72]),

$$\deg (T_n, H_{m,n}, 0) = \deg (T_{n,m}, H_{m,n} \cap E_m, 0),$$

where $E_m = \text{span} (e_{m-1}, e_m)$ with orientation induced by the order of the basis elements, and $T_{n,m}$ is equal to $T_n$ restricted to $E_m$. Let $U_m$ be the translation on $E_m$ given by

$$U_m (x) = x + m e_m.$$ 

Then it follows (see Appendix II) that

$$\deg (T_{n,m}, H_{m,n} \cap E_m, 0) = \deg (T_{n,m} U_m, U_m^{-1} (H_{m,n} \cap E_m), 0).$$

Now

$$U_m^{-1} (H_{m,n} \cap E_m) = \{ x \in E_m : \| x \| < \frac{1}{2} \}$$

and

$$T_{n,m} U_m (\alpha_{m-1} e_{m-1} + \alpha_m e_m) = \gamma_{m-1} e_{m-1} + \gamma_m e_m,$$
where
\[ \gamma_{m-1} = \varepsilon_m \prod_{i=1}^{k_m} (\alpha_m - a_i \alpha_{m-1}). \]

and
\[ \gamma_m = \prod_{i=1}^{k_m} (\alpha_m - b_i \alpha_{m-1}). \]

Hence, as shown by Cronin [3, pp. 38–40],
\[ \deg (T_{n,m} U_m, U_m^{-1} (H_{m,n} \cap E_m), 0) = \varepsilon_m k_m. \]

Also \( T_n \) is the identity on \( H_{1,n} \) and \( 0 \not\in \text{cl} (H_{1,n}) \) so \( \deg (T_n, H_{1,n}, 0) = 0. \) Hence
\[ \deg (T_n, G_n, 0) = \sum_{m=1}^{n} \deg (T_n, H_{m,n}, 0) = \sum_{m=1}^{n} \varepsilon_m k_m = t_n \]
since \( t_1 = 0. \) Thus
\[ D (T, G, 0) = [t_n] = [s_n]. \]

**Appendix I.**
1. We show that \( \text{cl} (H_m) \cap \text{cl} (H_n) = \emptyset \) for \( m \neq n. \) If \( x \in \text{cl} (H_m) \) and \( y \in \text{cl} (H_n), \) then
\[ \|x - y\| \geq \|me_m - ne_n\| - \|me_m - x\| - \|ne_n - y\| \]
\[ \geq (m^2 + n^2)^{1/2} - \frac{1}{2} - \frac{1}{2} \geq 2^{1/2} - 1 > 0. \]

Hence \( x \neq y \) and \( \text{cl} (H_m) \cap \text{cl} (H_n) = \emptyset. \)

2. It also follows that \( \text{cl} (G) = \text{cl} (\bigcup_{m=1}^{\infty} H_m) = \bigcup_{m=1}^{\infty} \text{cl} (H_m). \) For if \( \{g_n\} \subseteq G \) is such that \( g_n \to x \) then there is an \( N \) such that \( \|g_n - g_m\| < (2^{1/2} - 1)/2 \) for all \( n, m \geq N. \) Hence there is a \( p \) such that \( g_n \in H_p \) for all \( n \geq N. \) Hence \( x \in \text{cl} (H_p) \) and \( \text{cl} (G) \subseteq \bigcup_{m=1}^{\infty} \text{cl} (H_m). \) Clearly
\[ \bigcup_{m=1}^{\infty} \text{cl} (H_m) \subseteq \bigcup_{m=1}^{\infty} \text{cl} (G), \]
so \( \text{cl} (G) = \bigcup_{m=1}^{\infty} \text{cl} (H_m). \)

3. We will now show that \( \text{cl} (H_m) \cap \text{span} (e_1, \ldots, e_n) = \emptyset \) for all \( m > n. \) If \( x \in \text{cl} (H_m) \) and \( y = \sum_{i=1}^{n} \alpha_i e_i \in \text{span} (e_1, \ldots, e_n) \) and \( m > n \) then
\[ \|x - y\| = \left\| me_m - \sum_{i=1}^{n} \alpha_i e_i \right\| = \left( m^2 + \sum_{i=1}^{n} \alpha_i^2 \right)^{1/2} \geq m > 0. \]

Hence \( x \neq y \) and \( \text{cl} (H_m) \cap \text{span} (e_1, \ldots, e_n) = \emptyset, \) for \( m > n. \)

**Appendix II.** Let \( D \) be an open bounded subset of \( R^n, f \) a continuous mapping from \( \text{cl} (D) \) to \( R^n \) and \( q \not\in f (\partial D). \) Let \( x_0 \in X \) be fixed. Let \( U \) be the translation defined by \( U (x) = x + x_0. \) We will show that
\[ \deg (fU, U^{-1} (D), q) = \deg (f, D, q). \]

By the product formula for the Brouwer degree [8, Theorem 3.20, p. 75]
\[ \text{deg} (fU, U^{-1}(D), q) = \sum_i \text{deg} (f, B_i, q) \text{deg} (U, U^{-1}(D), p_i), \]

where \( B_i \) are the bounded components of \( R^n \setminus U(\partial U^{-1}(D)) = R^n \setminus \partial D \) and \( p_i \in B_i \). But

\[ \text{deg} (U, U^{-1}(D), p_i) = \sum \text{sign} |U'(x)|, \]

where \( |U'(x)| \) is the determinant of the Jacobian matrix of \( U \) at \( x \), and the sum is over all points \( x \) in \( U^{-1}(p_i) \cap U^{-1}(D) \), i.e. \( x = p_i - x_0, p_i \in B_i \cap D \). Now \( U(x + h) - U(x) = h \) for all \( x, h \in \mathbb{R}^n \), so \( U'(x) \) is the identity matrix and \( |U'(x)| = 1 \). Hence,

\[ \text{deg} (fU, U^{-1}(D), q) = \sum_i \text{deg} (f, B_i, q), \]

where the sum is over the bounded components of \( D \). Thus by the sum formula for the Brouwer degree (see [3, Theorem 6.8, p. 32] or [8, Theorem 3.16.5, p. 72]), \( \text{deg} (fU, U^{-1}(D), q) = \text{deg} (f, D, q) \).

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References


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