HOMOTOPY EQUIVALENCES IN EQUIVARIANT TOPOLOGY

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Abstract. Homomorphisms up to homotopy (higher homotopies that is) are generalized for the equivariant category. Homotopy equivalences have an inverse in this new category.

Introduction. In equivariant topology the notion of a homotopy equivalence presents a problem. Strictly within the equivariant category, homotopy equivalence seems to be too limited a concept, e.g. the unit interval \( I \) (acting on itself as an \( H \)-space) is not of the same equivariant homotopy type as \( \{1\} \) as a \( \{1\} \)-space.

Some of the rather general tools used in homotopy theory of topological groups and \( H \)-spaces (e.g. studying classifying spaces) can also be used in equivariant homotopy theory. In this paper we use \( G_\infty \)-maps between \( G \)-spaces (as defined in 1.4, and similar to \( H_\infty \)-maps between \( H \)-spaces) to study a new notion of homotopy equivalence. Roughly speaking, if \( X \) and \( \bar{X} \) are \( G \)-spaces and if \( f : X \to \bar{X} \) is a \( G \)-equivariant map and also an ordinary homotopy equivalence, then there exists a sequence of maps \( g_n : X \times (I \times G)^n \to \bar{X} \) forming a \( G_\infty \)-map such that \( f \) and the maps \( \{g_n\} \) form a pair of \( G_\infty \)-homotopy equivalences.

The complete theorem is stated in §2. The proof includes the proof of the corresponding theorem on \( H \)-spaces as stated in [2, Theorem 4.1] or in [1] as Proposition 1.17. We hope to convince the reader that this proof is not as messy as it is described by the authors of [1] on p. 13.

1. Definitions.

Definition 1.1. An \( H \)-space \( G \) is a topological space with a continuous multiplication \( \mu \). We assume that \( \mu \) is strictly associative. No unit element is needed.

Definition 1.2. A topological space \( X \) is called a \( G \)-space, if \( G \) acts on \( X \) in a continuous and in an associative manner.

Multiplication and actions will be denoted by the usual juxtaposition.

Definition 1.3. An \( H_\infty \)-map \( h \) from \( G \) to \( \bar{G} \) of length \( r^2 \) is a sequence of continuous maps \( h_n : G \times (I \times G)^n \to \bar{G} \) such that

Presented to the Society, August 22, 1975; received by the editors August 13, 1974 and, in revised form, November 4, 1975.


1 C. N. Lee, A. Wasserman (see [5]), and T. Petrie have relatively simple examples of pairs of manifolds \( X \) and \( Y \) with \( S^1 \)-actions and an \( S^1 \)-map \( f : X \to Y \) such that \( f \) is an ordinary homotopy equivalence. But there is no equivariant map \( g : Y \to X \) which is a homotopy equivalence.

2 We will mention the length of maps only if it is essential to the context.
Definition 1.4. Let \( X \) be a \( G \)-space, \( \overline{X} \) be a \( \overline{G} \)-space and let \( h \) be an \( H_\infty \)-map from \( G \) to \( \overline{G} \) of length \( r \). A \( G_\infty \)-map \( f \) from \( X \) to \( \overline{X} \) of length \( r \) associated to \( h \) is a sequence of maps \( f_n : X \times (I_r \times G)^n \to \overline{X} \) such that

\[
f_n(x, t_1, g_1, \ldots, t_n, g_n)
= \begin{cases}
  f_{n-1}(x, t_1, \ldots, g_{i-1} g_i, \ldots, t_n, g_n), & t_i = 0, n \geq 0, \\
  f_{i-1}(x, t_1, g_1, \ldots, g_{i-1}) h_{n-i}(g_i, \ldots, g_n), & t_i = r,
\end{cases}
\]

where \( g_0, \ldots, g_n \in G \) and \( t_1, \ldots, t_n \in I_r = [0, r] \subset \mathbb{R} \). If \( r = 0 \), the map \( h_0 \) is an ordinary homomorphism.

Composition of \( G_\infty \)-maps is defined the same way as for \( H_\infty \)-maps. If \( \sigma = 1 \)

\[
(f \circ f)_n(x, t_1, g_1, \ldots, t_n, g_n)
= f_{n-1}(x, t_1, g_1, \ldots, t_n, g_n),
\]

which is the standard composition of homotopies. If \( n > 1 \) we have to form the composite of \( 2^n \) maps, each of which is defined on one of the \( 2^n \) rectangular boxes obtained from \( I_{r+s}^n \) be partitioning it with the hyperplanes \( t_i = r \).

Definition 1.5. Let \( X_1 \) be a \( G_1 \)-space, \( X_2 \) a \( G_2 \)-space, and \( X_3 \) a \( G_3 \)-space. Let \( h \) be an \( H_\infty \)-map of length \( r \) from \( G_1 \) and \( G_2 \) and \( \overline{h} \) an \( H_\infty \)-map of length \( s \) from \( G_2 \) to \( G_3 \). Also let \( f \) be a \( G_\infty \)-map of length \( r \) from \( X_1 \) to \( X_2 \) associated to \( h \), and let \( \overline{f} \) be a \( G_\infty \)-map of length \( s \) from \( X_2 \) to \( X_3 \) associated to \( \overline{h} \). We define \( (\overline{f} \circ f)_n : X_1 \times (I_{r+s} \times G_1)^n \to X_3 \) by

\[
(\overline{f} \circ f)_n(x, t_1, g_1, \ldots, t_n, g_n)
= \begin{cases}
  \overline{f}_n(f_i, t_1, g_1, \ldots, t_i, g_i), & 0 \leq t_1 \leq r, \\
  f_n(f_0, x, t_1 - r, h_0(y)), & r \leq t_1 \leq r + s,
\end{cases}
\]

Composition formulas for \( H_\infty \)-maps can also be found in [2], [3] and [4].

We leave it to the reader to verify that the maps \( (\overline{f} \circ f)_n \) form a \( G_\infty \)-maps associated to \( ((\overline{f} \circ f)_n) \). The composition of \( G_\infty \)-maps is associative and together with the \( G \)-spaces they form a category.

Definition 1.6. Let \( h^0 \) and \( h^1 \) be \( H_\infty \)-maps from \( G \) to \( \overline{G} \), and let \( f^0 \) and \( f^1 \) be \( G_\infty \)-maps from \( X \) to \( X \) associated to \( h^0 \) and \( h^1 \) respectively. If \( h^t (0 \leq t \leq 1) \) is a family of \( H_\infty \)-maps constituting an \( H_\infty \)-homotopy between
A family \( f^t \) (\( 0 \leq t \leq 1 \)) of \( G_\infty \)-maps from \( X \) to \( X \) associated with \( h^t \) is called a \( G_\infty \)-homotopy between \( f^0 \) and \( f^1 \).

**Definition 1.7.** A \( G_\infty \)-map \( f \) from \( X \) to \( X \) associated with the \( H_\infty \)-map \( h \) from \( G \) to \( \overline{G} \) is called a \( G_\infty \)-homotopy equivalence if there exists an \( H_\infty \)-map \( \overline{h} \) from \( \overline{G} \) to \( G \) and a \( G_\infty \)-map \( \overline{f} \) from \( X \) to \( X \) such that the respective compositions are \( G_\infty \)-homotopic (and \( H_\infty \)-homotopic respectively) to \( \text{id}_X \) and \( \text{id}_X \) associated to \( \text{id}_G \) and \( \text{id}_{\overline{G}} \) respectively. The \( G_\infty \)-map \( \overline{f} \) associated with \( \overline{h} \) is called a homotopy inverse of \( f \) associated with \( h \).

**2. Theorems and proofs.**

**Theorem.** Let \( X \) be a \( G \)-space and \( X \) be a \( G \)-space. If \( h \) is an \( H_\infty \)-map from \( G \) to \( \overline{G} \) such that \( h_0 \) is an ordinary homotopy equivalence, and \( f \) is a \( G_\infty \)-map from \( X \) to \( X \) associated with \( h \) such that \( f_0 \) is an ordinary homotopy equivalence, then \( f \) is a \( G_\infty \)-homotopy equivalence.

**Remark.** We construct both an inverse \( \overline{h} \) to \( h \) and an inverse \( \overline{f} \) to \( f \) associated with \( h \). However, the construction of \( \overline{f} \) works for any inverse of \( h \), and also for every homotopy inverse of \( f_0 \).

Two special cases are of importance.

**Corollary.** If \( X \) and \( X \) are \( G \)-spaces and \( f: X \rightarrow X \) is a \( G \)-equivariant map as well as an ordinary homotopy equivalence, then \( f \) is a \( G_\infty \)-homotopy equivalence associated to \( 1_G \).

**Corollary.** If \( H \) and \( \overline{H} \) are \( H \)-spaces and \( h: H \rightarrow \overline{H} \) is a strict homomorphism (or an \( H_\infty \)-map) as well as an ordinary homotopy equivalence, then \( h \) is an \( H_\infty \)-homotopy equivalence (see [2, Theorem 4.1] and [1]).

The second corollary is obtained by choosing \( X = H \) and \( X = \overline{H} \).

**Proof.** We will construct an \( H_\infty \)-map \( \overline{h}: \overline{G} \rightarrow G \) which is an \( H_\infty \)-homotopy inverse to \( h \), and a \( G_\infty \)-map \( \overline{f}: X \rightarrow X \) associated to \( \overline{h} \) which is a \( G_\infty \)-homotopy inverse to \( f \). The construction is by induction.

\( n = 0 \): Let \( f_0: X \rightarrow X \) be an ordinary homotopy inverse to \( f_0 \) and \( k_0: X \times I \rightarrow X \) be such that

\[
k_0(x, t) = \begin{cases} x & \text{for } t = 0, \\ f_0k_0(x) & \text{for } t = 1, \end{cases} \quad x \in X.
\]

Then, since \( f_0 \) and \( f_0 \) induce isomorphisms on the relevant homotopy classes of maps (compare e.g. [2, p. 205]) we can choose \( \overline{k}_0: \overline{X} \times I \rightarrow \overline{X} \) such that

\[
\overline{k}_0(\overline{x}, t) = \begin{cases} \overline{x} & \text{for } t = 0, \\ f_0f_0(\overline{x}) & \text{for } t = 1 \end{cases},
\]

in such a manner that there exists a map \( u_0: X \times I^2 \rightarrow X \) with

\[
\begin{align*}
u_0(x, 0, s_2) &= f_0k_0(x, s_2), \\
u_0(x, 1, s_2) &= \overline{k}_0(f_0(x), s_2)
\end{align*}
\]

\( s_2 \in I, x \in X, \)

and
\[ u_0(x, s_1, 1) = f_0 f_0 f_0(x) \quad \text{for all } s_1 \in I, \ x \in X. \]
\[ u_0(x, s_1, 0) = f_0(x) \]

We will call \( f_0 \) together with \((k_0, \overline{k}_0)\) and \( u_0 \) a compatible homotopy inverse to \( f_0 \).

Similarly we choose a compatible homotopy inverse \( \overline{h}_0 \) together with homotopies \((l_0, \overline{l}_0)\) and \( \nu_0 \) for the map \( h_0 \) (compare [2, p. 205]). We notice that this second choice is independent of the choice of \( f_0 \).

**Induction Hypothesis.** Assume we already constructed \((f_0, \ldots, f_{n-1})\) as the first \( n \) functions of a \( G_\infty \)-homotopy inverse for \( f \) associated with \((h_0, \ldots, \overline{h}_{n-1})\), the \( n \) functions of an \( H_\infty \)-homotopy inverse of \( h \). Furthermore assume that we constructed these functions as compatible inverses to \((f_0, \ldots, f_{n-1})\) and \((h_0, \ldots, h_{n-1})\). For convenience sake we assume that all maps of \( f \) and \( h \) have length one. Also \( f_0, \ldots, f_{n-1} \) and \( h_0, \ldots, \overline{h}_{n-1} \) are assumed to have length one. The homotopies \( k_0, \ldots, k_{n-1}, \overline{k}_0, \ldots, \overline{k}_{n-1}, l_0, \ldots, l_{n-1}, \overline{l}_0, \ldots, \overline{l}_{n-1} \) are to be of length two. The maps \( u_0, \ldots, u_{n-1} \) and \( \nu_0, \ldots, \nu_{n-1} \) are to have length three. These last maps shall satisfy the obvious boundary conditions as described in 1.3 and 1.4, in addition to the compatibility conditions.

Now we will construct \( u_n : X \times (I_3 \times G)^n \times I^2 \to X \). (The maps \( f_n, k_n \) and \( \overline{k}_n \) will be defined in the process of this construction.) For our construction by induction \( u_n \) shall have the following properties.

1. \( u_n | X \times (I_3 \times G)^n \times I \times \{0\} = (\overline{e}_1 \circ f \circ e_1) \times \text{id}_I \).

Here \( e_1 \) and \( \overline{e}_1 \) stands for the identity morphism of \( X \) and \( X \) respectively with length 1. We observe that this part of \( u_n \) is already known.

2. \( u_n | X \times (I_3 \times G)^n \times I \times \{1\} = (f \circ \overline{f} \circ f) \times \text{id}_I \).

The composition \((f \circ \overline{f} \circ f)\) contains the map \( f_n \) only if \( 1 \leq t_i \leq 2 \) (for \( i = 1, \ldots, n \)). For these values of \( t_i \) we have

\[(f \circ \overline{f} \circ f)(x, t_1, g_1, \ldots, g_n) = f_0 f_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)).\]

The rest of \((f \circ \overline{f} \circ f)\) uses \( \overline{f}_k \) with \( 1 \leq k \leq n \). Because of this and the boundary properties of \( f_n \) we know this part of \( u_n \) except when \( 1 < t_i < 2, i = 1, \ldots, n \).

3. \( u_n | X \times (I_3 \times G)^n \times \{0\} \times I = (\overline{e}_{1-s_2} \circ f \circ k_{1+s_2}) \times \text{id}_I \).

\( \overline{e}_{1-s_2} \) stands for the identity morphism of \( X \) with length \( 1 - s_2 \), and \( k_{1+s_2} \) stands for the homotopy \( k \) with length \( 1 + s_2 \), where \( s_2 \) is the second coordinate of \( I \times I \). From the composition rule for \( G_\infty \)-maps we see that this restriction of \( u_n \) is known except for \( 1 < t_i < 2 \) when \( s_2 = 1 \), and \( 2 - s_2 < t_i < 3 \) when \( 0 < s_2 < 1 \) (\( i = 1, \ldots, n \)), i.e. when

\[ u_n(x, t_1, g_1, \ldots, t_n, g_n, 0, s_2) \]
\[ = \begin{cases} 
 f_0 f_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)), & s_2 = 1, \\
 f_0 k_{1+s_2} n(x, t_1, \ldots, t_n, g_n, s_2), & 0 < s_2 < 1. 
\end{cases} \]
which is known except for $1 < t_i < 2$ when $s_2 = 1$, and $0 < t_i < 1 + s_2$ when $0 < s_2 < 1$, i.e. when

$$u_n(x, t_1, g_1, \ldots, t_n, g_n, 1, s_2) = \begin{cases} f_0(x), & 0 < s_2 < 1, \\ f_0(x), & s_2 = 1, \end{cases}$$

(5) $u_n|X \times \partial(I_3 \times G)^n \times I^2$ is defined by $u_0, \ldots, u_{n-1}$ and $v_0, \ldots, v_{n-1}$ according to the properties of $G_\infty$- and $H_\infty$-maps as described in 1.3 and 1.4. Hence this part of $u_n$ is known.

To find all of $u_n$, we are going to use the following consequence of the homotopy extension property:

**Lemma.** If $A \subset X$ has the homotopy extension property and if $r: Y \times I \to F_2$ are homotopy equivalences, then if $f: A \to Y_1$ is a map such that $r \circ fx$ has an extension $g_2 : X \to Y_2$, we know that $f_1$ has an extension $g_1 : A \to Y_1$ by the homotopy extension property.

We observe that the extension problem resulting from (3) is homeomorphic to extending a map, which is known on all of $F \times (\varepsilon_{\infty} \times G)^n$ with the exception of one face of $Y \times (\varepsilon_{\infty} \times G)^n$. We use $K_{1+s_2}$ to obtain $K_{1+s_2} : X \times (\varepsilon_{\infty} \times G)^n \times \{1\} \times I \to A$. For $s_2 = 1$ we see that $K_{1+s_2}$ provides an extension of $f_0 \circ f_0$ and $\partial(1,2] \times G)^n \to X \times (\varepsilon_{\infty} \times G)^n$. To obtain the extension of $\partial f_0^n : X \times (\varepsilon_{\infty} \times G)^n \to X$ we use the following lemma.

**Lemma.** Given a homotopy equivalence $a: A \to A$ a map $\partial c: A \times I \to B$, and a map $w: A \times I \to B$ that extends $\partial c(a \times 1)$, then $\partial c$ extends to a map $c: A \times I \to B$ such that $c(a \times 1)$ is homotopic to $w$ keeping the boundary fixed.

Since on $X \times \partial(1,2] \times G)^n \times \{1\} \times \{1\}$ we have

$$K_{1+s_2}(x, t_1, g_1, \ldots, t_n, g_n, 1, 1) = f_n(f_0(x), t_1, h_0(g_1), \ldots, t_n, h_0(g_n)),$$

we obtain that $K_{1+s_2}|X \times (\varepsilon_{\infty} \times G)^n \times \{1\} \times \{1\}$ is homotopic to $f_n \circ f_0 \times (1 \times h_0)$ relative to the boundary $X \times \partial(1,2] \times G)^n \times \{1\} \times \{1\}$. We combine $K_{1+s_2}$ with such a homotopy to obtain $k_{1+s_2}$, with $k_{1+s_2}|X \times (\varepsilon_{\infty} \times G)^n \times \{1\} \times \{1\} = f_n \circ f_0 \times (1 \times h_0)^n$.

We now know $u_n$ except on $X \times (I_3 \times G)^n \times I^2$ and the part of $X \times (I_3 \times G)^n \times \{1\} \times I$ with $0 < t_i < 1 + s_2$ and $0 < s_2 < 1$. This extension problem is homeomorphic to the problem of extending a map which is known on all of $Y \times \partial I^{n+2}$ with the exception of one face to all of $Y \times I^{n+2}$. Let $U_n$ be such an extension. We have to use the second lemma once more to obtain $U_{1+s_2}$. Again $K_{1+s_2}(f_0 \times 1 \times h_0 \times \cdots \times h_0)$ is homotopic to the right restriction of $U_n$ leaving the boundary fixed. We alter $U_n$ by this homotopy and obtain $U_{n}$.

The construction of $v_n, h_n, l_n$, and $l_n$ is completely analogous and is left to the reader.
REFERENCES


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