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MEAN VALUE THEOREMS FOR ARITHMETIC FUNCTIONS SIMILAR TO EULER'S PHI-FUNCTION

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ABSTRACT. This paper establishes mean value results for multiplicative functions satisfying $f(p^e) = p^{e-1}f(p)$ as well as certain conditions on the differences $f(p) - p$.

If ϕ is Euler's function, it is well known [8] that

$$\lim_{x\to\infty} x^{-1} \sum_{n\leqslant x} \frac{n}{\phi(n)} = \zeta(2)\zeta(3)/\zeta(6),$$

where ζ denotes the Riemann zeta function. Also, if $N(x)$ denotes the number of values of n for which $\phi(n) \leqslant x$, then it is known [1], [4], [5] that

$$\lim_{x\to\infty} x^{-1}N(x) = \zeta(2)\zeta(3)/\zeta(6).$$

The purpose of this paper is to generalize these two theorems to arithmetical functions similar to Euler's function. The most important properties of ϕ in this regard are that it is a multiplicative function such that $\phi(p^e) = p^{e-1}\phi(p)$ for all primes p and all positive integers e and that $\phi(p)$ is "close to" p.

Our first theorem is proved using a simple lemma on Dirichlet series. It is well known and easy to prove. A proof may be found in a paper by D. G. Kendall and R. A. Rankin [7, Lemma 4].

LEMMA 1. *Let $d_1, d_2, \ldots$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} d_n/n$ is absolutely convergent. Then if*

$$\sum_{m=1}^{\infty} c_m m^{-s} = \sum_{m=1}^{\infty} m^{-s} \sum_{n=1}^{\infty} d_n n^{-s} \quad (\operatorname{Re} s > 1),$$

we have

$$\lim_{x\to\infty} x^{-1} \sum_{m\leqslant x} c_m = \sum_{n=1}^{\infty} d_n n^{-1}.$$

THEOREM 1. *Let k be a positive integer. Let f be a nonvanishing multiplicative function such that $f(p^e) = p^{e-1}f(p)$ for all primes p and all positive integers e. Suppose that the series*

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$$\sum_p \frac{p - f(p)}{pf(p)} \tag{1}$$

is absolutely convergent. Then

$$\sum_{n \leqslant x;(n,k)=1} \frac{n}{f(n)} \sim A_k \frac{\phi(k)}{k} x, \tag{2}$$

and

$$\sum_{n \leqslant x;(n,k)=1} \frac{1}{f(n)} \sim A_k \frac{\phi(k)}{k} \log x, \tag{3}$$

where

$$A_k = \prod_{p|k} \left(1 + \frac{1}{f(p)} - \frac{1}{p}\right).$$

PROOF. To prove (2) we consider the Dirichlet series

$$\begin{aligned}
\sum_{n=1;(n,k)=1}^{\infty} \frac{n}{f(n)} n^{-s} &= \prod_{p|k} \left(1 + \frac{p}{f(p)p^s} + \frac{p^2}{f(p^2)p^{2s}} + \cdots\right) \\
&= \prod_{p|k} \left(1 + \frac{p}{f(p)}\left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)\right) \\
&= \zeta(s) \prod_{p|k} \left(1 - \frac{1}{p^s}\right) \prod_{p|k} \left(1 + \frac{p - f(p)}{p^s f(p)}\right).
\end{aligned}$$

We have absolute convergence for $\operatorname{Re} s > 1$. Let

$$g(s) = \prod_{p|k} \left(1 - \frac{1}{p^s}\right) \prod_{p|k} \left(1 + \frac{p - f(p)}{p^s f(p)}\right).$$

The absolute convergence of (1) shows that the Dirichlet series for g converges absolutely for $s = 1$. Thus, by Lemma 1, we have

$$\lim_{x \to \infty} x^{-1} \sum_{n \leqslant x;(n,k)=1} \frac{n}{f(n)} = g(1) = A_k \frac{\phi(k)}{k}.$$

The result (3) follows from (2) by partial summation. □

The result (2) was obtained by H. Delange [2], [3] under somewhat different hypotheses. Instead of the hypothesis that f is nonvanishing he required the stronger hypothesis that $|f(n)| \geqslant n$ for all n. Instead of the hypothesis that the series (1) is absolutely convergent he used the weaker hypothesis that it is merely convergent.

If we put $f(n) = \phi(n)$ in Theorem 1, we have

$$\sum_{n \leqslant x;(n,k)=1} \frac{n}{\phi(n)} \sim B_k x$$

and

$$\sum_{n \leqslant x;(n,k)=1} \frac{1}{\phi(n)} \sim B_k \log x, \tag{4}$$

where

$$\begin{aligned} B_k &= \frac{\phi(k)}{k} \prod_{p|k} \left(1 + \frac{1}{p-1} - \frac{1}{p}\right) \\ &= \prod_{p|k} \left(\frac{p^2 - 2p + 1}{p^2 - p + 1}\right) \prod_p \left(1 + \frac{1}{p(p-1)}\right) \\ &= \prod_{p|k} \left(\frac{p^2 - 2p + 1}{p^2 - p + 1}\right) \zeta(2)\zeta(3)/\zeta(6). \end{aligned}$$

E. Landau [8] proved (4) with $k = 1$. E. C. Titchmarsh [10] first proved (4) for arbitrary k using complex analysis. T. Estermann [6] and E. Landau [9] made improvements by elementary methods.

COROLLARY 1. *Let f be a nonvanishing multiplicative function such that $f(p^e) = p^{e-1}f(p)$ for all primes p and all positive integers e. Suppose $p - f(p) = O(p(\log p)^{-\varepsilon})$ for some $\varepsilon > 0$. Then we have* (2) *and* (3).

PROOF. If $p - f(p) = O(p(\log p)^{-\varepsilon})$, then $f(p) > p/2$ for large p. Thus

$$|(p - f(p))/pf(p)| = O(p^{-1}(\log p)^{-\varepsilon}).$$

Thus the series (1) is absolutely convergent. □

If f is a positive multiplicative function such that $f(p)$ tends to infinity as p tends to infinity and $f(p^e) = p^{e-1}f(p)$, it is clear that $f(p^e)$ tends to infinity as p^e tends to infinity. Then $f(n)$ tends to infinity with n. Let $B(x)$ denote the number of positive integers n such that $f(n) \leqslant x$. If the series (1) is absolutely convergent we know that (2) holds for $k = 1$, so that $n/f(n)$ has the mean value A, where

$$A = \prod_p \left(1 + \frac{1}{f(p)} - \frac{1}{p}\right).$$

Thus we would expect the number of n for which $f(n) \leqslant x$ to be about the number of n for which $n/A \leqslant x$, which is $[Ax]$. Thus we would expect that $x^{-1}B(x)$ tends to A as x tends to infinity. We will prove that this is true under a slightly stronger hypothesis.

THEOREM 2. *Let f be a positive multiplicative function such that $f(p^e) = p^{e-1}f(p)$ for all primes p and all positive integers e. Suppose that the series*

$$\sum_p \frac{p - f(p)}{pf(p)} \log(p + f(p)) \tag{5}$$

is absolutely convergent. Let $B(x)$ be the number of positive integers n with $f(n) \leqslant x$. Then

$$\lim_{x\to\infty} x^{-1}B(x) = A.$$

PROOF. Note that, for $\operatorname{Re} s \geqslant 1$,

$$\begin{aligned}\left|\frac{1}{p^s} - \frac{1}{f(p)^s}\right| &= \left|s\int_{\log p}^{\log f(p)} e^{-us}\,du\right| \\ &\leqslant |s|\cdot\left|\int_{\log p}^{\log f(p)} e^{-u}\,du\right| = |s|\cdot\left|\frac{1}{p} - \frac{1}{f(p)}\right| \\ &= |s|\cdot\left|\frac{p - f(p)}{pf(p)}\right|.\end{aligned}$$

Thus, for $\operatorname{Re} s > 1$, we can write

$$\begin{aligned}\sum_{n=1}^{\infty} f(n)^{-s} &= \prod_p (1 + f(p)^{-s} + f(p^2)^{-s} + \cdots) \\ &= \prod_p \left(1 + \frac{1}{f(p)^s}(1 + p^{-s} + p^{-2s} + \cdots)\right) = \zeta(s)P(s),\end{aligned} \tag{6}$$

where

$$P(s) = \prod_p \left(1 + \frac{1}{f(p)^s} - \frac{1}{p^s}\right).$$

The product defining $P(s)$ converges absolutely for $\operatorname{Re} s \geqslant 1$, so P is continuous for $\operatorname{Re} s \geqslant 1$.

It is easy to see that the convergence of the series (5) implies that $f(p)$ tends to infinity with p.

Since $(1 + f(p)^{-s} - p^{-s})$ tends to one as p tends to infinity for $\operatorname{Re} s \geqslant 1$, the absolute convergence of the series

$$\sum_p (1 + f(p)^{-s} - p^{-s})^{-1}\left(\frac{\log p}{p^s} - \frac{\log f(p)}{f(p)^s}\right)$$

follows from the absolute convergence of (5) and the inequality

$$\begin{aligned}\left|\frac{\log p}{p^s} - \frac{\log f(p)}{f(p)^s}\right| &= \left|\int_{\log p}^{\log f(p)} (1 - us)e^{-us}\,du\right| \\ &\leqslant \{1 + |s|\max(\log p, \log f(p))\}\left|\int_{\log p}^{\log f(p)} e^{-u}\,du\right| \\ &\leqslant \{1 + |s|\log(p + f(p))\}|1/p - 1/f(p)|,\end{aligned}$$

where we have assumed p large enough so that $f(p) > 1$. Thus logarithmic differentiation shows that P is differentiable at $s = 1$.

Now we note that for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} f(n)^{-s} = \sum_{m=1}^{\infty} b_m c_m^{-s} = \sum_{m=1}^{\infty} b_m e^{-s\log c_m}, \tag{7}$$

where $0 < c_1 < c_2 < \cdots$ are the values of f, and b_m is the number of times f assumes the value c_m.

Combining (6) and (7) we have

$$(8)\quad \sum_{m=1}^{\infty} b_m e^{-s\log c_m} - \frac{P(1)}{s-1} = \left(\zeta(s) - \frac{1}{s-1}\right)P(s) + \frac{P(s)-P(1)}{s-1} = h(s),$$

say, where $h(s)$ is continuous for $\operatorname{Re} s \geqslant 1$ in view of the differentiability of P at $s = 1$. Now we let $C(x) = B(e^x)$ and note that, for $\operatorname{Re} s > 1$,

$$(9)\qquad \sum_{c_m > 1} b_m e^{-s\log c_m} = \int_0^\infty e^{-su}\,dC(u) = s\int_0^\infty e^{-su}C(u)\,du - C(0).$$

Combining (8) and (9) we see that

$$\int_0^\infty e^{-su}C(u)\,du - \frac{P(1)}{s-1} = \frac{1}{s}\left\{C(0) - P(1) - \sum_{c_m \leqslant 1} b_m e^{-s\log c_m} + h(s)\right\}.$$

Now the Wiener-Ikehara Theorem, whose proof is in [11], implies that

$$\lim_{x\to\infty} x^{-1}B(x) = \lim_{x\to\infty} e^{-x}C(x) = P(1),$$

which is the desired conclusion. □

COROLLARY 2. *Let f be a positive multiplicative function such that $f(p^e) = p^{e-1}f(p)$ for all primes p and all positive integers e. Suppose that $p - f(p) = O(p(\log p)^{-\varepsilon})$ for some $\varepsilon > 0$. Then the conclusion of Theorem 2 holds.*

If we let $f = \phi$ in Theorem 2 or Corollary 2 we see that

$$\lim_{x\to\infty} x^{-1}N(x) = \prod_p \left(1 + \frac{1}{p-1} - \frac{1}{p}\right) = \zeta(2)\zeta(3)/\zeta(6),$$

where $N(x)$ is the number of n with $\phi(n) \leqslant x$. Erdös [5] first proved that $x^{-1}N(x)$ has a finite limit as x tends to infinity. Once the existence of the limit is known it is easy to evaluate.

REFERENCES

1. P. T. Bateman, *The distribution of values of the Euler function*, Acta Arith. **21** (1972), 329–345. MR **46** #1730.

2. H. Delange, *Sur les fonctions arithmétiques multiplicatives*, Ann. Sci. École Norm. Sup. (3) **78** (1961), 273–304. MR **30** #72.

3. ———, *Application de la méthode du crible à l'étude des valeurs moyennes de certaines fonctions arithmétiques*, Séminaire Delange-Pisot (mimeographed lecture notes), Paris, 1962.

4. R. E. Dressler, *A density which counts multiplicity*, Pacific J. Math. **34** (1970), 371–378. MR **42** #5940.

5. P. Erdös, *Some remarks on Euler's ϕ-function and some related problems*, Bull. Amer. Math. Soc. **51** (1945), 540–544. MR **7**, 49.

6. T. Estermann, *On an asymptotic formula due to Titchmarsh*, J. London Math. Soc. **6** (1931), 250–251.

7. D. G. Kendall and R. A. Rankin, *On the number of Abelian groups of a given order*, Quart. J. Math. Oxford Ser. **18** (1947), 197–208. MR **9**, 226.

8. E. Landau, *Ueber die zahlentheoretische Funktion* $\phi(n)$ *und ihre Beziehung zum Goldbachschen Satz*, Göttinger Nachr. 1900, 177–186.

9. E. Landau, *On a Titchmarsh-Estermann sum*, J. London Math. Soc. **11** (1936), 242–245.

10. E. C. Titchmarsh, *A divisor problem*, Rend. Circ. Mat. Palmero **54** (1930), 414–429.

11. D. V. Widder, *The Laplace transform*, Princeton Math. Ser., vol. 6, Princeton Univ. Press, Princeton, N.J., 1941. MR **3**, 232.

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