ISOMETRIC MULTIPLIERS AND ISOMETRIC ISOMORPHISMS OF $l_1(S)$

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Abstract. Let $S$ be a commutative semigroup and $\Omega(S)$ the multiplier semigroup of $S$. It is shown that $T$ is an isometric multiplier of $l_1(S)$ if and only if there exists an invertible element $\sigma \in \Omega(S)$ and a complex number $\lambda$ of unit modulus such that $T(\alpha) = \lambda \sum_{x \in S} \alpha(x) a_{\sigma(x)}$ for each $\alpha = \sum_{x \in S} a(x) \delta_x \in l_1(S)$.

Also, if $S_1$ and $S_2$ are commutative semigroups, and $L$ is an isometric isomorphism of $l_1(S_1)$ into $l_1(S_2)$, then it is proved that there exist a semicharacter $\chi$, $|\chi(x)| = 1$ for all $x \in S_1$, and an isomorphism $i$ of $S_1$ onto $S_2$ such that $L(\alpha) = \sum \chi(x) a(x) b_{i(x)}$ for each $\alpha = \sum_{x \in S_1} a(x) \delta_x \in l_1(S_1)$.

1. Introduction. Let $S$ be a commutative semigroup. As usual, $l_1(S)$ denotes the Banach space of all complex functions $\alpha: S \to \mathbb{C}$ such that $\|\alpha\| = \sum_{x \in S} |\alpha(x)|$ is finite. Letting $\delta_x \in l_1(S)$ represent the point mass at $x \in S$, an arbitrary element $\alpha$ of $l_1(S)$ is of the form $\alpha = \sum_{x \in S} a(x) \delta_x$, $a(x) \in \mathbb{C}$ for all $x \in S$; in fact, $a(x) \neq 0$ for at most countably many elements of $S$. The linear space $l_1(S)$ becomes a commutative Banach algebra under the convolution product

$$\alpha * \beta = \sum_{x \in S} \sum_{u,v; uv = x} a(u) \beta(v) \delta_x,$$

where $\alpha$ as above and $\beta = \sum_{x \in S} \beta(x) \delta_x \in l_1(S)$. Further, $\hat{S}$ denotes the set of all semicharacters on $S$, that is, the set of all bounded nonzero functions $\chi: S \to \mathbb{C}$ such that $\chi(xy) = \chi(x) \chi(y)$ for all $x, y \in S$. For a fuller treatment of $l_1(S)$, consult [3].

Given a semigroup $S$, define $\Omega(S)$ to be the set of all functions $\sigma: S \to S$ having the property that $\sigma(xy) = x \sigma(y)$ for all $x, y \in S$. Under the operation of composition of functions, $\Omega(S)$ is a semigroup and is called the multiplier semigroup of $S$. Note that $\Omega(S)$ always has an identity $e$, the identity function on $S$. Throughout this paper assume that $\Omega(S)$ is commutative: a sufficient condition for the commutativity of $\Omega(S)$ is that $l_1(S)$ is semisimple. For weaker conditions implying commutativity and a more extensive discussion of $\Omega(S)$, consult [4, Proposition 4.1].

A bounded linear operator $T: l_1(S) \to l_1(S)$ is called a multiplier of...
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If \( T(\alpha \ast \beta) = \alpha \ast T(\beta) \) for all \( \alpha, \beta \in l_1(S) \). The set of multipliers of \( l_1(S) \) is a commutative Banach algebra of operators under operator norm and is denoted \( \mathfrak{M}(l_1(S)) \). An operator \( T \in \mathfrak{M}(l_1(S)) \) is an isometric multiplier of \( l_1(S) \) if \( T \) is a one-to-one mapping of \( l_1(S) \) onto \( l_1(S) \) with the property that \( \|T(\alpha)\| = \|\alpha\| \) for all \( \alpha \in l_1(S) \). For a general discussion of multipliers of Banach algebras and isometric multipliers, see [5].

Each element \( \tau = \sum_{\sigma \in \Omega(S)} \tau(\sigma)\delta_{\sigma} \in l_1(\Omega(S)) \) determines a multiplier \( T_\sigma \) of \( l_1(S) \) as follows: defining \( T_\sigma \) first at each point mass \( \delta_x \) of \( l_1(S) \) by \( T_\sigma(\delta_x) = \sum_{\sigma \in \Omega(S)} \tau(\sigma)\delta_{\sigma(x)}, x \in S \), extend \( T_\sigma \) linearly to the subspace \( P \) of \( l_1(S) \) spanned by the set of point masses; \( T_\sigma \) becomes a bounded operator on \( l_1(S) \) by observing that \( T_\sigma \) is bounded on \( P \) and that \( P \) is dense in \( l_1(S) \) [4, Proposition 4.2]. For each \( \sigma \in \Omega(S) \), \( T_\sigma \) will denote the multiplier \( T_{\delta_{\sigma}} \).

If \( G \) is a locally compact group, denote by \( L_1(G) \) the group (Banach) algebra of Haar integrable functions on \( G \) under convolution multiplication. It is a well-known result that the isometric multipliers of \( L_1(G) \) consist of scalar multiples of translation operators [7, Theorem 3]. The next section of this paper, §2, is devoted to a discussion of the isometric multipliers of \( l_1(S) \). It is shown that if \( T \) is an isometric multiplier of \( l_1(S) \), then there exist \( \sigma \in \Omega(S) \) and \( \lambda \in \mathbb{C}, |\lambda| = 1 \), such that \( T = \lambda T_\sigma \).

Moreover, it is also known that an isometric isomorphism of two group algebras induces an isomorphism of the underlying groups [6]. In §3, Theorem 3.1, an analogous result is obtained for \( l_1 \)-algebras.

2. Isometric multipliers of \( l_1(S) \).

Proposition 2.1. Let \( \sigma \in \Omega(S) \). Then

(a) \( T_\sigma \) is an isometric multiplier of \( l_1(S) \) if and only if \( \sigma \) is one-to-one and onto, and

(b) \( \sigma \) is one-to-one and onto if and only if \( \sigma \) is invertible in \( \Omega(S) \).

Proof. (a) Let \( T_\sigma \) be an isometric multiplier. Then the one-to-oneness of \( T_\sigma \) implies that \( T_\sigma(\delta_x) = T_\sigma(\delta_y) \) if and only if \( x = y, x, y \in S \); hence, \( \sigma(x) = \sigma(y) \) if and only if \( x = y \), and thus \( \sigma \) is one-to-one. Similarly, \( \sigma \) is onto.

If, now, \( \sigma \) is one-to-one and onto, then \( T_\sigma(\delta_x) = \delta_{\sigma(x)}, x \in S \), shows that \( T_\sigma \) is a one-to-one map of the set of point masses \( \{\delta_x : x \in S\} \) onto itself. Hence, if \( \alpha = \sum \sigma(x)\delta_x \in l_1(S) \), then \( T_\sigma(\alpha) = \sum_{x \in S} \alpha(x)\delta_{\sigma(x)} \) is such that \( \|\alpha\| = \|T_\sigma(\alpha)\| \).

(b) If \( \sigma \) is one-to-one and onto, then for \( x \in S \), define \( \sigma^{-1}(x) = y \) if and only if \( \sigma(y) = x \). Then \( \sigma^{-1}(xz) = y \Rightarrow xz = \sigma(y) \), and \( \sigma^{-1}(z) = r \Rightarrow z = \sigma(r) \), which implies that \( \sigma(xr) = x\sigma(r) = \sigma(y) \) and hence \( xr = y \) since \( \sigma \) is one-to-one, or \( \sigma^{-1}(xz) = y = x\sigma^{-1}(z) \); that is \( \sigma^{-1} \in \Omega(S) \).

Conversely, if there exists \( \sigma^{-1} \in \Omega(S) \) such that \( \sigma\sigma^{-1} = e \), then \( \sigma(x) = \sigma(y) \) implies \( x = y \) (showing \( \sigma \) is one-to-one), and for a given \( z \in S \), \( \sigma(\sigma^{-1}(z)) = z \) (showing \( \sigma \) is onto). □

Define
I = \{ \sigma \in \Omega(S): \sigma \text{ is one-to-one and onto} \}
= \{ \sigma \in \Omega(S): \sigma \text{ is invertible in } \Omega(S) \}.

Observe that I is never empty since e \in I. The next theorem shows that I determines the set of isometric multipliers of \( L_1(S) \).

**THEOREM 2.1.** Let \( T \in \mathcal{M}(L_1(S)) \). Then \( T \) is an isometric multiplier of \( L_1(S) \) if and only if \( T = \lambda T_\sigma \) for some complex number \( \lambda \) of unit modulus and some \( \sigma \in I \).

**Proof.** Let \( T \) be an isometric multiplier of \( L_1(S) \). Then \( T \) maps the unit ball of \( L_1(S) \) onto the unit ball of \( L_1(S) \), and, in particular, \( T \) maps the extreme points of the unit ball onto the extreme points of the unit ball. Now, the collection of extreme points of the unit ball of \( L_1(S) \) is the set \( \{ \lambda \delta_x: \lambda \in \mathbb{C}, |\lambda| = 1, x \in S \} \) [2, p. 81]. Hence, let us suppose that \( T(\delta_x) = \lambda_x \delta_x, x \in S \).

Then, for \( x, y \in S \),
\[
\lambda_{xy} \delta_{xy} = T(\delta_{xy}) = \delta_x \ast T(\delta_y) = \delta_x \ast \lambda_y \delta_y = \lambda_y \delta_{xy},
\]
and also
\[
T(\delta_{xy}) = \delta_y \ast T(\delta_x) = \delta_y \ast \lambda_x \delta_x = \lambda_x \delta_{yx}.
\]
Thus,
\[
(xy)' = xy = x'y \quad \text{and} \quad \lambda_{(xy)} = \lambda_y = \lambda_x \quad \text{for all } x, y \in S.
\]

Hence, there exists a unique complex number \( \lambda \) of unit modulus such that \( T(\delta_x) = \lambda \delta_x \) for all \( x \in S \). Moreover, define a function \( \sigma: S \to S \) by \( \sigma(x) = x', x \in S \). From above, the fact that \( \sigma(xy) = x\sigma(y) = y\sigma(x) \) for all \( x, y \in S \) implies that \( \sigma \in \Omega(S) \). Therefore, \( T = \lambda T_\sigma, \sigma \in I \); by Proposition 2.1; and the implication is proved.

The converse follows immediately from Proposition 2.1. \( \square \)

Part (b) of the following proposition shows that in many cases there may be very few isometric multipliers, indeed. Part (a) is an instance of Wendel's result for \( L_1(G) \), where \( G \) is a discrete group.

**Proposition 2.2.** (a) If \( S \) has an identity \( e \), then \( I = \{ x \in S: x \text{ is invertible in } S \} \). In particular, if \( S \) is a group, \( I = S \).

(b) If \( S \) is an idempotent semigroup, then \( I = \{ e \} \) and the only isometric multiplier is the identity multiplier.

**Proof.** (a) If \( S \) has an identity \( e \), then \( S = \Omega(S) \) and the result follows from Proposition 2.1(b).

(b) Let \( \sigma \in \Omega(S), \sigma \neq e \); hence, there exist \( x, y \in S \) such that \( x \neq y \) and \( \sigma(x) = y \). Then \( xy = x\sigma(x) = \sigma(x^2) = \sigma(x) = y \) implies that \( y = y' = \sigma(y) = \sigma(xy) = \sigma(x) = \sigma(y) \). Thus, \( \sigma \) is not one-to-one and, therefore, is not in \( I \). \( \square \)
3. Isometric isomorphisms of \( l_1(S) \). Let \( S_1 \) and \( S_2 \) be two commutative semigroups, and let \( \Gamma = \{ \chi \in S_1^\prime : |\chi(x)| = 1 \text{ for all } x \in S_1 \} \).

**Theorem 3.1.** For each \( \chi \in \Gamma \) and for each isomorphism \( i : S_1 \rightarrow S_2 \) of \( S_1 \) onto \( S_2 \), the linear operator \( L : l_1(S_1) \rightarrow l_1(S_2) \), defined by \( L(\alpha) = \sum \chi(x)\alpha(x)\delta_{i(x)} \) for each \( \alpha = \sum_{x \in S_1} \alpha(x)\delta_x \in l_1(S_1) \), is an isometric isomorphism of \( l_1(S_1) \) onto \( l_1(S_2) \). Conversely, if \( L \) is an isometric isomorphism of \( l_1(S_1) \) onto \( l_1(S_2) \), then there exist \( \chi \in \Gamma \) and an isomorphism \( i \) of \( S_1 \) onto \( S_2 \) such that \( L(\alpha) = \sum_{x \in S_1} \chi(x)\alpha(x)\delta_{i(x)} \) for each \( \alpha = \sum_{x \in S_1} \alpha(x)\delta_x \in l_1(S_1) \).

**Proof.** Suppose \( L \) is an isometric isomorphism of \( l_1(S_1) \) onto \( l_1(S_2) \). As in the proof of Theorem 2.1, \( L \) maps the extreme points of the unit ball of \( l_1(S_1) \) onto the extreme points of the unit ball of \( l_1(S_2) \); say, \( L(\delta_x) = \lambda_x \delta_{x'} \), \( x \in S_1 \), \( \lambda_x \in C \), \( |\lambda_x| = 1 \). Then for \( x, y \in S_1 \), \( \lambda_y \delta_x \delta_{x'y} = \lambda_x \delta_x \delta_{y} = L(\delta_x) \hat{} \delta_{x'y} \). Also note that \( x, y \in S_1 \), \( x \neq y \), implies \( x' \neq y' \): for if \( L(\delta_x) = \lambda_x \delta_{x'} \), then for any \( \lambda \in C \), \( |\lambda| = 1 \), \( L(\lambda \delta_{x'} / \lambda_{x'}) = \lambda \delta_{x} \); hence, the one-to-oneness of \( L \) and the fact that \( \lambda \delta_{x}/ \lambda_x \neq \delta_{y} \) for any \( \lambda \in C \) imply that \( L(\delta_{x'}) \neq \lambda \delta_{x} \) for all \( \lambda \in C \), \( |\lambda| = 1 \).

Define a map \( i : S_1 \rightarrow S_2 \) by \( i(x) = x' \), \( x \in S_1 \); then \( i \) is an isomorphism of \( S_1 \) onto \( S_2 \) since \( x \neq y \) implies \( x' \neq y' \) and \( L \) maps the extreme points of \( l_1(S_1) \) onto the extreme points of \( l_1(S_2) \).

Finally, define a map \( \chi : S_1 \rightarrow C \) by \( \chi(x) = \lambda_x \), \( x \in S_1 \). Then the fact that \( \chi(xy) = \lambda_{xy} = \lambda_x \lambda_y = \chi(x)\chi(y) \), \( x, y \in S_1 \), and \( |\chi(x)| = |\lambda_x| = 1 \) for all \( x \in S_1 \) show that \( \chi \in \Gamma \). \( \square \)

**Theorem 3.2.** Let \( L \) be an isometric isomorphism of \( l_1(S_1) \) onto \( l_1(S_2) \). Then

(a) \( L \) induces an isomorphism \( \tilde{L} \) of \( \mathfrak{M}(l_1(S_1)) \) onto \( \mathfrak{M}(l_1(S_2)) \), and

(b) \( L \) maps the isometric multipliers of \( l_1(S_1) \) onto the isometric multipliers of \( l_1(S_2) \). Consequently, if \( T_0 \) is an isometric multiplier of \( l_1(S_1) \), then there exists an invertible multiplier \( \phi \in \Omega(S_2) \) and a complex number \( \lambda_\phi \) of unit modulus such that \( \lambda_\phi T_0 L^{-1} = \phi \).

**Proof.** (a) Define \( L : \mathfrak{M}(l_1(S_1)) \rightarrow \mathfrak{M}(l_1(S_2)) \) by \( \tilde{L}(T) = LTL^{-1} \), \( T \in \mathfrak{M}(l_1(S_1)) \). To see that \( \tilde{L}(T) \) actually belongs to \( \mathfrak{M}(l_1(S_2)) \), observe that because \( L, L^{-1} \) and \( T \) are bounded linear operators, \( \tilde{L}(T) \) is a bounded linear operator. Also, the boundedness and linearity of \( \tilde{L}(T) \) necessitate only verifying its multiplicative behavior on the set of point masses: for \( x, y \in S_2 \)

\[
LTL^{-1}(\delta_{xy}) = LTL^{-1}(\delta_{x}) \cdot LTL^{-1}(\delta_{y})
\]

\[
= L(L^{-1}(\delta_{x}) \cdot T(L^{-1}(\delta_{y})))
\]

\[
= \delta_{x} \cdot LTL^{-1}(\delta_{y}).
\]

Thus, \( \tilde{L}(T) \in \mathfrak{M}(l_1(S_2)) \), as claimed. That \( \tilde{L} \) is one-to-one and onto is clear.

(b) From Theorem 2.1 and (a), it is sufficient to show that \( \lambda_\phi T_0 L^{-1} \) is an
isometric multiplier of $l_1(S_2)$ for each isometric multiplier $T_o$ of $l_1(S_1)$. Let $T_o$ be an isometric multiplier of $l_1(S_1)$. Since $L$ maps point masses of $l_1(S_1)$ to scalar multiples of point masses of $l_1(S_2)$ in a one-to-one manner, and since $T_o$ behaves similarly on $l_1(S_1)$, it is only necessary to compute the norm of $LT_o$ at an arbitrary point mass of $l_1(S_2)$. That is, if $x' \in S_2$, it suffices to show that $\|LT_o L^{-1}(\delta_{x'})\| = 1$. However, observe that for a given $x' \in S_2$ there exist $x \in S_1$ and $\lambda_\chi \in \mathbb{C}$, $|\lambda_\chi| = 1$, such that $L(\lambda_\chi \delta_{\sigma(x)}) = \lambda_\chi \delta_{x'}$. Then $T_o(\lambda_\chi \delta_{\sigma(x)}) = \lambda_\chi \delta_{\sigma(x)} \chi$ implies that $L(\lambda_\chi \delta_{\sigma(x)}) = \lambda_\chi \lambda_\chi \delta_{\sigma(x)} \chi$ for some $(\lambda_\chi)' \in \mathbb{C}$, $|(\lambda_\chi)'| = 1$, $\sigma(x)' \in \Omega(S_2)$; hence, $\|LT_o L^{-1}(\delta_{x'})\| = |\lambda_\chi(\lambda_\chi)'| = 1$. Thus, the existences of $\lambda_\chi \in \mathbb{C}$ and $T_o$ defined in the statement of the theorem follow from Theorem 2.1.

It should be noted that there is a more general statement of Theorem 3.2(a) in [1, Theorem 1].

Although an isometric isomorphism of $l_1$-algebras can be extended to an isomorphism of the respective multiplier algebras, it is not true that an isomorphism of multiplier algebras induces an isomorphism of the underlying $l_1$-algebras. Let $S_1$ be the set of negative integers under the operation of maximum multiplication; $\Omega(S_1) = S_1 \cup \{e\}$; that is, $\Omega(S_1)$ is obtained from $S_1$ by adjoining an identity. Let $S_2 = S_1 \cup \{e\}$; clearly, $\Omega(S_2) = S_2$. $\mathfrak{M}(l_1(S_1)) = l_1(\Omega(S_1))$ since $l_1(S_1)$ has a bounded approximate identity [4], and $\mathfrak{M}(l_1(S_2)) = l_1(S_2)$. Certainly, $\mathfrak{M}(l_1(S_1))$ is isomorphic (isometrically) to $\mathfrak{M}(l_1(S_2))$, but $l_1(S_1)$ is not isomorphic to $l_1(S_2)$.

References


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