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ISOMETRIC MULTIPLIERS AND ISOMETRIC ISOMORPHISMS OF $l_1(S)$

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ABSTRACT. Let S be a commutative semigroup and $\Omega(S)$ the multiplier semigroup of S. It is shown that T is an isometric multiplier of $l_1(S)$ if and only if there exists an invertible element $\sigma \in \Omega(S)$ and a complex number λ of unit modulus such that $T(\alpha) = \lambda \sum_{x \in S} \alpha(x)\delta_{\sigma(x)}$ for each $\alpha = \sum_{x \in S} \alpha(x)\delta_x \in l_1(S)$.

Also, if S_1 and S_2 are commutative semigroups, and L is an isometric isomorphism of $l_1(S_1)$ into $l_1(S_2)$, then it is proved that there exist a semicharacter χ, $|\chi(x)| = 1$ for all $x \in S_1$, and an isomorphism i of S_1 onto S_2 such that $L(\alpha) = \sum \chi(x)\alpha(x)\delta_{i(x)}$ for each $\alpha = \sum_{x \in S_1} \alpha(x)\delta_x \in l_1(S_1)$.

1. **Introduction.** Let S be a commutative semigroup. As usual, $l_1(S)$ denotes the Banach space of all complex functions $\alpha: S \to \mathbf{C}$ such that $\|\alpha\| = \sum_{x \in S} |\alpha(x)|$ is finite. Letting $\delta_x \in l_1(S)$ represent the point mass at $x \in S$, an arbitrary element α of $l_1(S)$ is of the form $\alpha = \sum_{x \in S} \alpha(x)\delta_x$, $\alpha(x) \in \mathbf{C}$ for all $x \in S$; in fact, $\alpha(x) \neq 0$ for at most countably many elements of S. The linear space $l_1(S)$ becomes a commutative Banach algebra under the convolution product

$$\alpha * \beta = \sum_{x \in S} \sum_{u,v;uv=x} \alpha(u)\beta(v)\delta_x,$$

where α is as above and $\beta = \sum_{x \in S} \beta(x)\delta_x \in l_1(S)$. Further, $\hat{S}$ denotes the set of all semicharacters on S, that is, the set of all bounded nonzero functions $\chi: S \to \mathbf{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. For a fuller treatment of $l_1(S)$, consult [3].

Given a semigroup S, define $\Omega(S)$ to be the set of all functions $\sigma: S \to S$ having the property that $\sigma(xy) = x\sigma(y)$ for all $x, y \in S$. Under the operation of composition of functions, $\Omega(S)$ is a semigroup and is called the multiplier semigroup of S. Note that $\Omega(S)$ always has an identity e, the identity function on S. Throughout this paper assume that $\Omega(S)$ is commutative: a sufficient condition for the commutativity of $\Omega(S)$ is that $l_1(S)$ is semisimple. For weaker conditions implying commutativity and a more extensive discussion of $\Omega(S)$, consult [4, Proposition 4.1].

A bounded linear operator $T: l_1(S) \to l_1(S)$ is called a multiplier of

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$l_1(S)$ if $T(\alpha * \beta) = \alpha * T(\beta)$ for all $\alpha, \beta \in l_1(S)$. The set of multipliers of $l_1(S)$ is a commutative Banach algebra of operators under operator norm and is denoted $\mathfrak{M}(l_1(S))$. An operator $T \in \mathfrak{M}(l_1(S))$ is an isometric multiplier of $l_1(S)$ if T is a one-to-one mapping of $l_1(S)$ onto $l_1(S)$ with the property that $\|T(\alpha)\| = \|\alpha\|$ for all $\alpha \in l_1(S)$. For a general discussion of multipliers of Banach algebras and isometric multipliers, see [5].

Each element $\tau = \sum_{\sigma \in \Omega(S)} \tau(\sigma)\delta_\sigma \in l_1(\Omega(S))$ determines a multiplier T_τ of $l_1(S)$ as follows: defining T_τ first at each point mass δ_x of $l_1(S)$ by $T_\tau(\delta_x) = \sum_{\sigma \in \Omega(S)} \tau(\sigma)\delta_{\sigma(x)}$, $x \in S$, extend T_τ linearly to the subspace P of $l_1(S)$ spanned by the set of point masses; T_τ becomes a bounded operator on $l_1(S)$ by observing that T_τ is bounded on P and that P is dense in $l_1(S)$ [4, Proposition 4.2]. For each $\sigma \in \Omega(S)$, T_σ will denote the multiplier T_{δ_σ}.

If G is a locally compact group, denote by $L_1(G)$ the group (Banach) algebra of Haar integrable functions on G under convolution multiplication. It is a well-known result that the isometric multipliers of $L_1(G)$ consist of scalar multiples of translation operators [7, Theorem 3]. The next section of this paper, §2, is devoted to a discussion of the isometric multipliers of $l_1(S)$. It is shown that if T is an isometric multiplier of $l_1(S)$, then there exist $\sigma \in \Omega(S)$ and $\lambda \in \mathbf{C}$, $|\lambda| = 1$, such that $T = \lambda T_\sigma$.

Moreover, it is also known that an isometric isomorphism of two group algebras induces an isomorphism of the underlying groups [6]. In §3, Theorem 3.1, an analogous result is obtained for l_1-algebras.

2. **Isometric multipliers of** $l_1(S)$.

PROPOSITION 2.1. *Let $\sigma \in \Omega(S)$. Then*

(a) *T_σ is an isometric multiplier of $l_1(S)$ if and only if σ is one-to-one and onto, and*

(b) *σ is one-to-one and onto if and only if σ is invertible in $\Omega(S)$.*

PROOF. (a) Let T_σ be an isometric multiplier. Then the one-to-oneness of T_σ implies that $T_\sigma(\delta_x) = T_\sigma(\delta_y)$ if and only if $x = y$, $x, y \in S$; hence, $\sigma(x) = \sigma(y)$ if and only if $x = y$, and thus σ is one-to-one. Similarly, σ is onto.

If, now, σ is one-to-one and onto, then $T_\sigma(\delta_x) = \delta_{\sigma(x)}$, $x \in S$, shows that T_σ is a one-to-one map of the set of point masses $\{\delta_x : x \in S\}$ onto itself. Hence, if $\alpha = \sum \alpha(x)\delta_x \in l_1(S)$, then $T_\sigma(\alpha) = \sum_{x \in S} \alpha(x)\delta_{\sigma(x)}$ is such that $\|\alpha\| = \|T_\sigma(\alpha)\|$.

(b) If σ is one-to-one and onto, then for $x \in S$, define $\sigma^{-1}(x) = y$ if and only if $\sigma(y) = x$. Then $\sigma^{-1}(xz) = y \Rightarrow xz = \sigma(y)$, and $\sigma^{-1}(z) = r \Rightarrow z = \sigma(r)$, which implies that $\sigma(xr) = x\sigma(r) = \sigma(y)$ and hence $xr = y$ since σ is one-to-one, or $\sigma^{-1}(xz) = y = x\sigma^{-1}(z)$; that is $\sigma^{-1} \in \Omega(S)$.

Conversely, if there exists $\sigma^{-1} \in \Omega(S)$ such that $\sigma\sigma^{-1} = e$, then $\sigma(x) = \sigma(y)$ implies $x = y$ (showing σ is one-to-one), and for a given $z \in S$, $\sigma(\sigma^{-1}(z)) = z$ (showing σ is onto). $\square$

Define

$$I = \{\sigma \in \Omega(S)\colon \sigma \text{ is one-to-one and onto}\}$$
$$= \{\sigma \in \Omega(S)\colon \sigma \text{ is invertible in } \Omega(S)\}.$$

Observe that I is never empty since $e \in I$. The next theorem shows that I determines the set of isometric multipliers of $l_1(S)$.

THEOREM 2.1. *Let $T \in \mathfrak{M}(l_1(S))$. Then T is an isometric multiplier of $l_1(S)$ if and only if $T = \lambda T_\sigma$ for some complex number λ of unit modulus and some $\sigma \in I$.*

PROOF. Let T be an isometric multiplier of $l_1(S)$. Then T maps the unit ball of $l_1(S)$ onto the unit ball of $l_1(S)$, and, in particular, T maps the extreme points of the unit ball onto the extreme points of the unit ball. Now, the collection of extreme points of the unit ball of $l_1(S)$ is the set $\{\lambda\delta_x\colon \lambda \in \mathbf{C}, |\lambda| = 1, x \in S\}$ [2, p. 81]. Hence, let us suppose that $T(\delta_x) = \lambda_{x'}\delta_{x'}$, $x \in S$. Then, for $x, y \in S$,

$$\lambda_{(xy)'}\delta_{(xy)'} = T(\delta_{xy}) = \delta_x * T(\delta_y) = \delta_x * \lambda_{y'}\delta_{y'} = \lambda_{y'}\delta_{xy'}$$

and also

$$T(\delta_{xy}) = \delta_y * T(\delta_x) = \delta_y * \lambda_{x'}\delta_{x'} = \lambda_{x'}\delta_{yx'}.$$

Thus,

$$(xy)' = xy' = x'y \quad \text{and} \quad \lambda_{(xy)'} = \lambda_{y'} = \lambda_{x'} \quad \text{for all } x, y \in S.$$

Hence, there exists a unique complex number λ of unit modulus such that $T(\delta_x) = \lambda\delta_{x'}$ for all $x \in S$. Moreover, define a function $\sigma\colon S \to S$ by $\sigma(x) = x'$, $x \in S$. From above, the fact that $\sigma(xy) = x\sigma(y) = y\sigma(x)$ for all $x, y \in S$ implies that $\sigma \in \Omega(S)$. Therefore, $T = \lambda T_\sigma$, $\sigma \in I$; by Proposition 2.1; and the implication is proved.

The converse follows immediately from Proposition 2.1. □

Part (b) of the following proposition shows that in many cases there may be very few isometric multipliers, indeed. Part (a) is an instance of Wendel's result for $L_1(G)$, where G is a discrete group.

PROPOSITION 2.2. (a) *If S has an identity e, then $I = \{x \in S\colon x$ is invertible in $S\}$. In particular, if S is a group, $I = S$.*

(b) *If S is an idempotent semigroup, then $I = \{e\}$ and the only isometric multiplier is the identity multiplier.*

PROOF. (a) If S has an identity e, then $S = \Omega(S)$ and the result follows from Proposition 2.1(b).

(b) Let $\sigma \in \Omega(S)$, $\sigma \neq e$; hence, there exist $x, y \in S$ such that $x \neq y$ and $\sigma(x) = y$. Then $xy = x\sigma(x) = \sigma(x^2) = \sigma(x) = y$ implies that $y = y^2 = y\sigma(x) = \sigma(xy) = \sigma(y)$. Thus, σ is not one-to-one and, therefore, is not in I. □

3. **Isometric isomorphisms of** $l_1(S)$. Let S_1 and S_2 be two commutative semigroups, and let $\Gamma = \{\chi \in \hat{S}_1 : |\chi(x)| = 1 \text{ for all } x \in S_1\}$.

THEOREM 3.1. *For each* $\chi \in \Gamma$ *and for each isomorphism* $i: S_1 \to S_2$ *of* S_1 *onto* S_2, *the linear operator* $L: l_1(S_1) \to l_1(S_2)$, *defined by* $L(\alpha) = \sum \chi(x)\alpha(x)\delta_{i(x)}$ *for each* $\alpha = \sum_{x \in S_1} \alpha(x)\delta_x \in l_1(S_1)$, *is an isometric isomorphism of* $l_1(S_1)$ *onto* $l_1(S_2)$. *Conversely, if* L *is an isometric isomorphism of* $l_1(S_1)$ *onto* $l_1(S_2)$, *then there exist* $\chi \in \Gamma$ *and an isomorphism* i *of* S_1 *onto* S_2 *such that* $L(\alpha) = \sum_{x \in S_1} \chi(x)\alpha(x)\delta_{i(x)}$ *for each* $\alpha = \sum_{x \in S_1} \alpha(x)\delta_x \in l_1(S_1)$.

PROOF. Suppose L is an isometric isomorphism of $l_1(S_1)$ onto $l_1(S_2)$. As in the proof of Theorem 2.1, L maps the extreme points of the unit ball of $l_1(S_1)$ onto the extreme points of the unit ball of $l_1(S_2)$; say, $L(\delta_x) = \lambda_x \delta_{x'}$, $x \in S_1$, $\lambda_x \in \mathbf{C}$, $|\lambda_x| = 1$. Then for $x, y \in S_1$, $\lambda_x \lambda_y \delta_{x'y'} = \lambda_x \delta_{x'} * \lambda_y \delta_{y'} = L(\delta_x) * L(\delta_y) = L(\delta_{xy}) = \lambda_{xy}\delta_{(xy)'}$ implies that $x'y' = (xy)'$ and $\lambda_x \lambda_y = \lambda_{xy}$. Also note that $x, y \in S_1$, $x \neq y$, implies $x' \neq y'$: for if $L(\delta_x) = \lambda_x \delta_{x'}$, then for any $\lambda \in \mathbf{C}$, $|\lambda| = 1$, $L(\lambda\delta_x/\lambda_x) = \lambda\delta_{x'}$; hence, the one-to-oneness of L and the fact that $\lambda\delta_x/\lambda_x \neq \delta_y$ for any $\lambda \in \mathbf{C}$ imply that $L(\delta_y) \neq \lambda\delta_{x'}$ for all $\lambda \in \mathbf{C}$, $|\lambda| = 1$.

Define a map $i: S_1 \to S_2$ by $i(x) = x'$, $x \in S_1$; then i is an isomorphism of S_1 onto S_2 since $x \neq y$ implies $x' \neq y'$ and L maps the extreme points of $l_1(S_1)$ onto the extreme points of $l_1(S_2)$.

Finally, define a map $\chi: S_1 \to \mathbf{C}$ by $\chi(x) = \lambda_x$, $x \in S_1$. Then the fact that $\chi(xy) = \lambda_{xy} = \lambda_x \lambda_y = \chi(x)\chi(y)$, $x, y \in S_1$, and $|\chi(x)| = |\lambda_x| = 1$ for all $x \in S_1$ show that $\chi \in \Gamma$. □

THEOREM 3.2. *Let* L *be an isometric isomorphism of* $l_1(S_1)$ *onto* $l_1(S_2)$. *Then*

(a) L *induces an isomorphism* $\overline{L}$ *of* $\mathfrak{M}(l_1(S_1))$ *onto* $\mathfrak{M}(l_1(S_2))$, *and*

(b) $\overline{L}$ *maps the isometric multipliers of* $l_1(S_1)$ *onto the isometric multipliers of* $l_1(S_2)$. *Consequently, if* T_σ *is an isometric multiplier of* $l_1(S_1)$, *then there exists an invertible multiplier* $\sigma' \in \Omega(S_2)$ *and a complex number* λ_σ *of unit modulus such that* $LT_\sigma L^{-1} = \lambda_\sigma T_{\sigma'}$.

PROOF. (a) Define $\overline{L}: \mathfrak{M}(l_1(S_1)) \to \mathfrak{M}(l_1(S_2))$ by $\overline{L}(T) = LTL^{-1}$, $T \in \mathfrak{M}(l_1(S_1))$. To see that $\overline{L}(T)$ actually belongs to $\mathfrak{M}(l_1(S_2))$, observe that because L, L^{-1} and T are bounded linear operators, $\overline{L}(T)$ is a bounded linear operator. Also, the boundedness and linearity of $\overline{L}(T)$ necessitate only verifying its multiplicative behavior on the set of point masses: for $x, y \in S_2$

$$\begin{aligned} LTL^{-1}(\delta_{xy}) &= LT(L^{-1}(\delta_x) * L^{-1}(\delta_y)) \\ &= L(L^{-1}(\delta_x) * T(L^{-1}(\delta_y))) \\ &= \delta_x * LTL^{-1}(\delta_y). \end{aligned}$$

Thus, $\overline{L}(T) \in \mathfrak{M}(l_1(S_2))$, as claimed. That $\overline{L}$ is one-to-one and onto is clear.

(b) From Theorem 2.1 and (a), it is sufficient to show that $LT_\sigma L^{-1}$ is an

isometric multiplier of $l_1(S_2)$ for each isometric multiplier T_σ of $l_1(S_1)$. Let T_σ be an isometric multiplier of $l_1(S_1)$. Since L maps point masses of $l_1(S_1)$ to scalar multiples of point masses of $l_1(S_2)$ in a one-to-one manner, and since T_σ behaves similarly on $l_1(S_1)$, it is only necessary to compute the norm of $\overline{L}(T_\sigma)$ at an arbitrary point mass of $l_1(S_2)$. That is, if $x' \in S_2$, it suffices to show that $\|LT_\sigma L^{-1}(\delta_{x'})\| = 1$. However, observe that for a given $x' \in S_2$ there exist $x \in S_1$ and $\lambda_x \in \mathbf{C}$, $|\lambda_x| = 1$, such that $L^{-1}(\delta_{x'}) = \lambda_x \delta_x$. Then $T_\sigma(\lambda_x \delta_x) = \lambda_x \delta_{\sigma(x)}$ implies that $L(\lambda_x \delta_{\sigma(x)}) = \lambda_x(\lambda_x)'\delta_{\sigma(x)'}$ for some $(\lambda_x)' \in \mathbf{C}$, $|(\lambda_x)'| = 1$, $\sigma(x)' \in \Omega(S_2)$; hence, $\|LT_\sigma L^{-1}(\delta_{x'})\| = |\lambda_x(\lambda_x)'| = 1$. Thus, the existences of $\lambda_\sigma \in \mathbf{C}$ and $T_{\sigma'}$ defined in the statement of the theorem follow from Theorem 2.1. $\square$

It should be noted that there is a more general statement of Theorem 3.2(a) in [1, Theorem 1].

Although an isometric isomorphism of l_1-algebras can be extended to an isomorphism of the respective multiplier algebras, it is not true that an isomorphism of multiplier algebras induces an isomorphism of the underlying l_1-algebras. Let S_1 be the set of negative integers under the operation of maximum multiplication; $\Omega(S_1) = S_1 \cup \{e\}$; that is, $\Omega(S_1)$ is obtained from S_1 by adjoining an identity. Let $S_2 = S_1 \cup \{e\}$; clearly, $\Omega(S_2) = S_2$. $\mathfrak{M}(l_1(S_1)) = l_1(\Omega(S_1))$ since $l_1(S_1)$ has a bounded approximate identity [4], and $\mathfrak{M}(l_1(S_2)) = l_1(S_2)$. Certainly, $\mathfrak{M}(l_1(S_1))$ is isomorphic (isometrically) to $\mathfrak{M}(l_1(S_2))$, but $l_1(S_1)$ is not isomorphic to $l_1(S_2)$.

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