STRONGLY EXPOSED POINTS IN WEAKLY COMPACT CONVEX SETS IN BANACH SPACES

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Abstract. A "purely geometric" proof of the Lindenstrauss-Troyanski result ([2], [6]) on strongly exposed points of weakly compact sets in Banach spaces is given.

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We consider a Banach-space $X$, $\| \|$. If $\xi \in X$ and if $\varepsilon > 0$, then $B(\xi, \varepsilon) = \{ x \in X; \| x - \xi \| < \varepsilon \}$. The closed convex hull of a set $A \subset X$, is denoted $\overline{c}(A)$. Let $C$ be a convex subset of $X$, then $C^e$ is the set of the extremal points of $C$. If $K$ is a convex, weakly compact, $B$ a convex, closed, bounded set of $X$ and $J$ a closed subinterval of $[0, 1]$, $(K, B, J)$ denotes the closed, convex set $\{(1 - t)k + tb; k \in K, b \in B, t \in J\}$.

We have to introduce a few geometrical definitions, referring to [4].

Suppose that $C$ is a nonempty, bounded, closed and convex subset of $X$. Let $M(C) = \sup \{ \| x \|; x \in C \}$. If $f \in X^*$ with $\| f \| = 1$, let $M(f, C) = \sup \{ f(x); x \in C \}$, and for each $\alpha > 0$, let $S(f, \alpha, C) = \{ x \in C; f(x) \geq M(f, C) - \alpha \}$. Such a set is called a "slice" of $C$. A point $\xi$ of $C$ is called "strongly exposed" if there exists $f \in X^*$ ($\| f \| = 1$) such that $\forall \varepsilon > 0, \exists \alpha > 0$ with $\xi \in S(f, \alpha, C) \subset B(\xi, \varepsilon)$. Let $S$ be the set of all $f \in X^*$ such that $\| f \| = 1$ and $f$ strongly exposes some point of $C$.

Proposition 1. If $K$ is a nonempty, convex, weakly-compact subset of $X$ and if $B$ is convex, closed and bounded with $K \cap B = \emptyset$, then the set $D = \overline{c}(K \cup B)$ has the following property: $\forall \varepsilon > 0, \exists \xi \in K$ such that $\xi \notin \overline{c}(D \setminus B(\xi, \varepsilon))$.

First we observe that it is sufficient to prove the proposition for $X$ separable.

Indeed, suppose $D$ does not have the required property. Then there exists $\varepsilon > 0$ with $\forall x \in K: x \in \overline{c}(D \setminus B(\xi, \varepsilon))$ and hence, $\forall x \in K, \exists A(x) \subset D$ with the properties: $x \in \overline{c}(A(x))$, $A(x) \cap B(\xi, \varepsilon) = \emptyset$ and $A(x)$ is countable. By induction we construct a sequence $(K_n, B_n)_n$ where $K_n$ is countable in $K$ and $B_n$ is countable in $B$.

Let $(K_0, B_0) = (\{ x \}, \emptyset)$, where $x$ is some element of $K$. Suppose we already found $(K_n, B_n)$.

Consider $\forall y \in \bigcup_{x \in K_n} A(x)$ an element $k_y$ in $K$ and $b_y$ in $B$, with $y \in \overline{c}(\{ k_y, b_y \})$.

Let

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197
Then \( K_{n+1} \) and \( B_{n+1} \) are still countable, which completes the construction.

Consider then \( K' = \bigcup_n K_n \), \( B' = \bigcup_n B_n \), \( D' = \overline{c}(K' \cup B') \) and the closed linear span \( X_0 \) of \( D' \).

Since \( V_n : c(A_n) \subseteq A_{n+1} \), \( A_n \) is convex and closed. The same holds for \( B' \). \( X_0 \) is a separable Banach-space wherein \( K' \) is nonempty, convex, weakly-compact and \( B' \) convex, closed and bounded. We show that \( D' \subset X_0 \) does not have the property mentioned in the proposition. Choose \( x' \in K' \) and take \( 0 < \delta < \varepsilon/2 \). There exist \( n \in \mathbb{N} \) and \( x \in K_n \) with \( \|x - x'\| < \delta \). We obtain:

\[
x \in \overline{c}(A(x)) \subset \overline{c}\left( \bigcup_{y \in A(x)} \overline{c}\left( (\{k_x, b_y\}) \setminus B(x, \varepsilon) \right) \right) \subset \overline{c}(D' \setminus B(x, \varepsilon)).
\]

Since this holds, \( \forall \delta (0 < \delta < \varepsilon/2) \), we have \( x' \in \overline{c}(D' \setminus B(x', \varepsilon/2)) \). \( D' \) does not have the property and we can restrict ourselves to the case of a separable Banach-space.

**Lemma.** \( D = \overline{c}(D^e \cup B) \).

**Proof.** Clearly \( E = \overline{c}(D^e \cup B) \subseteq D \). For the reverse inclusion it is sufficient to prove that \( K \subseteq E \). Suppose that \( x_0 \in K \setminus E \). Then \( \exists u \in X^* \) such that \( u(x_0) > \sup\{u(x); x \in D^e \cup B\} > \sup\{u(x); x \in B\} \). Let \( \alpha = \sup\{u(x); x \in K\} \), and let \( y \in D \). Then \( y = \lambda x_1 + (1 - \lambda)x_2 \) with \( x_1 \in K \), \( x_2 \in B \), \( \lambda \in [0, 1] \). It follows that

\[
u(y) = \lambda u(x_1) + (1 - \lambda)u(x_2) \leq \lambda u(x_1) + (1 - \lambda)u(x_0)
\]

This shows that \( \sup\{u(y); y \in D\} = \alpha \), and, since \( u(x_2) < u(x_0) \leq \alpha, u(y) = \alpha \) implies that \( \lambda = 1 \) or \( y \in K \). Let \( D_1 = \{ y \in D; u(y) = \alpha \} \). Then \( D_1 \) is a supporting set of \( D \), and \( D_1 \subset K \). Since \( D_1 \) is weakly compact, it contains an extreme point \( z \). Then \( z \in D^e \), but since \( u(z) = \alpha \), \( z \notin (D^e \cup B) \). This contradiction establishes that \( K \subseteq E \) or \( D = E \).

**Proof of Proposition 1.** The proof is a modification of a proof in Namioka [3]. We assume that \( X \) is separable. By the lemma, \( A = D^e \cup B \neq \emptyset \). Since \( A \subset X \) the weak closure \( \overline{A} \) of \( A \) is a Baire space relative to the weak topology. Since \( X \) is separable, there is a sequence \( \{x_n\} \) in \( X \) such that
Note that each $\overline{B}(x_n, \varepsilon/4)$ is weakly closed. Hence there is a weakly open set $N$ in $X$ such that $N \cap \overline{A} \neq \emptyset$ and $N \cap \overline{A} \subseteq \overline{B}(x_n, \varepsilon/4)$ for some $n$. Then clearly $\text{diam} (N \cap \overline{A}) \leq \varepsilon/2$. Let

$$D_1 = \bar{c}(N \cap \overline{A}) \quad \text{and} \quad D_2 = \bar{c}(\overline{(A \setminus N)} \cup B).$$

Then we observe that $D_1$ is weakly compact and $\text{diam} D_1 \leq \varepsilon/2$.

Since $A$ is weakly dense in $A$, $N \cap A \neq \emptyset$. Fix $\xi \in N \cap A \subseteq K$. Then $\xi \notin D_2$. For if $\xi \in D_2$, then $\xi \in \bar{c}(K \setminus N) \cup B \subseteq \overline{c}(K \setminus N), B,[0,1])$. Since $\xi \in \overline{D}$ and $\xi \notin B$, we have $\xi \in \bar{c}(K \setminus N)$. It then follows from the Krein-Milman theorem that $\xi \in K \setminus N$, because $K \setminus N$ is weakly compact. This contradicts $\xi \in N \cap A$.

Because $D = \bar{c}(D \setminus B)$, $D = \bar{c}(D_1 \cup D_2) = (D_1, D_2, [0,1])$. Let $C = (D_1, D_2, [\varepsilon/5d, 1])$, where $d = \text{diam} D$. Then $C$ is a closed convex subset of $D$.

If $\xi \in C$, then $\xi = (1 - \lambda)x_1 + \lambda x_2$, where $x_1 \in D_1$, $x_2 \in D_2$ and $\lambda \in [\varepsilon/5d, 1]$. Since $\xi$ is extreme, this implies that $\xi \in D_2$, which is impossible. Hence $\xi \notin C$. Let $y_1, y_2 \in D \setminus C$. Then $y_i = (1 - \lambda_i)x_1 + \lambda_i x_2$, where $x_1 \in D_1$, $x_2 \in D_2$ and $\lambda_i \in [0, \varepsilon/5d]$ ($i = 1, 2$). We then have:

$$\|y_1 - y_2\| \leq \|x_1 - x_2\| + \lambda_1 \|x_1 - x_2\| + \lambda_2 \|x_1^2 - x_2^2\|$$

$$\leq \varepsilon/2 + \varepsilon d/5d + \varepsilon d/5d = 9\varepsilon/10.$$

Since $\xi \in D \setminus C$ and $\text{diam} (D \setminus C) < \varepsilon$, it follows that $D \setminus C \subseteq B(\xi, \varepsilon)$. Therefore $D \setminus B(\xi, \varepsilon) \subseteq C$ and $\bar{c}(D \setminus B(\xi, \varepsilon)) \subseteq C$. Thus $\xi \notin \bar{c}(D \setminus B(\xi, \varepsilon))$ and the proof is complete.

**Remark.** The fact that $X, \|\|$ is complete is not used and the assertion is still true when $X$ is only a normed space.

**Proposition 2.** Let $C$ be a convex and weakly-compact subset of $X$. If $S(f, \alpha, C)$ is a slice of $C$, then $\forall \varepsilon > 0: \exists ! g \in X^*$, $\exists ! \beta > 0$ such that $S(g, \beta, C)$ is a slice of $C$ with diameter $\leq \varepsilon$, $S(g, \beta, C) \subseteq S(f, \alpha, C)$ and $\|f - g\| \leq \varepsilon$.

**Proof.** The proof is a modification of a proof in Phelps [4]. By translation we may assume that $0 \in H = \{x \in X; f(x) = M(f, C) - \alpha\}$. Hence $H = f^{-1}(0)$ and $\alpha = M(f, C) > 0$. We may also assume that $\varepsilon < \min(1, \alpha)$. Choose $\lambda$ so that $\lambda > 2M(C)/\varepsilon$, and let $K = S(f, \alpha/2, C)$ and $B = H \cap \overline{B}(0, \lambda)$. Since $K \cap B = \emptyset$ and $K \neq \emptyset$, we may apply Proposition 1 to $K$ and $B$.

Let $D = \bar{c}(K \cup B)$. Then $\exists ! \xi \in K$ such that $\xi \notin \bar{c}(D \setminus B(\xi, \varepsilon/2)).$ If $x \in B$, then $\|x - \xi\| \geq |f(x) - \xi| \geq \alpha/2 > \varepsilon/2.$ Therefore $B \subseteq \bar{c}(D \setminus B(\xi, \varepsilon/2)).$ By the separation theorem, $\exists ! g \in X^*$ such that $\|g\| = 1$ and $g(\xi) > \sup \{g(x); x \in D \setminus B(\xi, \varepsilon/2)\} \geq 0.$ Since $M(g, C) \geq g(\xi)$, we may write sup $\{g(x); x \in D \setminus B(\xi, \varepsilon/2)\} = M(g, C) - 2\beta$ with $\beta > 0$.

Then $S(g, \beta, C) \subseteq B(\xi, \varepsilon/2),$ and hence $\text{diam} S(g, \beta, C) \leq \varepsilon.$ Suppose that $f(x) = 0$ and $\|x\| \leq 1$. Then $\lambda x \in B$, and hence $\lambda g(x) < g(\xi)$ or $g(x) < \lambda^{-1} g(\xi).$ By Lemma 2 of [1], this implies that either $\|f - g\| \leq 2\lambda^{-1} g(\xi)$
$2\lambda^{-1}M(C) < \varepsilon$ or $\|f + g\| \leq 2\lambda^{-1}g(\xi)$. If the second possibility occurs, then $g(\xi) < (f + g)(\xi) \leq \|f + g\| \leq 2\lambda^{-1}g(\xi)M(C) < g(\xi)e$, which implies that $1 < \varepsilon$. But we assumed $\varepsilon < 1$. Therefore $\|f - g\| < \varepsilon$.

**Theorem.** Let $C$ be a convex and weakly-compact subset of $X$. Then $C$ is the closed convex hull of its strongly exposed points and the set $S$, defined in the introduction, is a dense $G_\delta$ subset of the unit sphere $\{ f \in X^*; \|f\| = 1 \}$ of $X^*$.

**Proof.** Referring to Lemma 7 of [4], the first assertion is a consequence of Proposition 2. We remark that it also follows from the second part of the theorem.

For $\varepsilon > 0$, let $U(\varepsilon)$ be the set of all $f \in X^*$ such that $\|f\| = 1$ and $\text{diam } S(f, \alpha, C) \leq \varepsilon$ for some $\alpha > 0$. Then $U(\varepsilon)$ is an open subset of the unit sphere of $X^*$. Indeed, suppose $f \in U(\varepsilon)$ and $\text{diam } S(f, \alpha, C) \leq \varepsilon$, then we verify that $S(g, \alpha/3, C) \subset S(f, \alpha, C)$ if $\|g\| = 1$ and $\|f - g\| < \alpha/3M(C)$. It is clear from Proposition 2 that $U(\varepsilon)$ is also dense there.

Since $S = \bigcap_{n=1}^{\infty} U(1/n)$, $S$ is a dense $G_\delta$ subset of the unit sphere by the Baire category theorem.

**Bibliography**


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