REMARKS ON $H(i)$ SPACES AND STRONGLY-CLOSED GRAPHS

LARRY L. HERRINGTON

Abstract. This paper gives characterizations of $H(i)$ spaces and strongly-closed graphs.

If $X$ and $Y$ topological spaces, we say a mapping $f: X \to Y$ has a strongly-closed graph if for each $(x,y) \not\in G(f) = \{(x,f(x)): x \in X\} \subset X \times Y$ there exist open sets $U$ and $V$ containing $x$ and $y$, respectively, such that $(U \times \text{cl}(V)) \cap G(f) = \emptyset$ (where $\text{cl}(V)$ denotes the closure of $V$). We say that a filterbase $F = \{A_\alpha \subset A: \alpha \in \Delta\}$ in $A \subset X$ $r$-converges\(^1\) to a point $a$ in $A$ ($F \to_r a$) if for every open $U \subset X$ containing $a$ there exists an $A_\alpha \subset \text{cl}(U)$ such that $A_\alpha \cap U \neq \emptyset$. In a like manner we define the concept of a net $\theta: D \to A \subset X$ $r$-converging to $a \in A$ ($\theta \to_r a$) and $r$-accumulating to a point $a \in A$ ($\theta \to_r a$). Of course, if $\theta: D \to A \subset X$ is a net in $A$, then the family $F(\theta) = \{\theta(T_b): b \in D\}$ (where $T_b = \{a \in D: a \succeq b\}$) is a filterbase in $A \subset X$ and it is routine to verify that (a) $F(\theta) \to_r a \in A$ if and only if $\theta \to_r a$ and (b) $F(\theta) \to_r a \in A$ if and only if $\theta \to_r a$. A mapping $f: X \to Y$ is weakly-continuous at $x \in X$ if for each open $V$ containing $f(x)$ there exists an open $U$ containing $x$ such that $f(U) \subset \text{cl}(V)$ [7]. It is noted in [5] that a mapping $f: X \to Y$ is weakly-continuous at $x \in X$ if and only if for each net $\{x_\alpha\}$ in $X$ converging to $x$, the net $\{f(x_\alpha)\} \to f(x)$. A subset $A \subset X$ is called regular-open (regular-closed) if it is the interior of its closure (resp., closure of its interior) [2, p. 92].

Characterizations of $C$-compact, $H$-closed, and minimal Hausdorff spaces are given in [4] and [5] by using the concepts of $r$-convergence, $r$-accumulation, strongly-closed graphs, and weakly-continuous functions. In this paper we further investigate strongly-closed graphs and give in addition some characterizations of feebly compact and $H(i)$ spaces.

1. Strongly-closed graphs and $H(i)$ subsets. We first characterize strongly-closed graphs in terms of filterbases.

Theorem 1.1. Let $f: X \to Y$ be given. Then $G(f)$ is strongly-closed if and only

\[^1\text{The concept of } r\text{-convergence and } r\text{-accumulation was first introduced by N. V. Veličko under the names of } \theta\text{-convergence and } \theta\text{-contact point respectively.}\]
if for each filterbase $F$ in $X$ such that $F$ converges to some $p$ in $X$ and $f(F)$ $r$-converges to some $q$ in $Y$, $f(p) = q$.

**Proof.** Suppose that $G(f)$ is strongly-closed and let $F = \{A_\alpha: \alpha \in \Delta\}$ be a filterbase in $X$ such that $F$ converges to $p$ and $f(F)$ $r$-converges to $q$. If $f(p) \neq q$, then $(p, q) \notin G(f)$. Thus there exist open sets $U \subset X$ and $V \subset Y$ containing $p$ and $q$, respectively, such that $(U \times \text{cl} (V)) \cap G(f) = \emptyset$. Since $F$ converges to $p$ and $f(F)$ $r$-converges to $q$, there exists an $A_\alpha \in F$ such that $A_\alpha \subset U$ and $f(A_\alpha) \subset \text{cl} (V)$). Consequently, $(U \times \text{cl} (V)) \cap G(f) \neq \emptyset$ which is a contradiction.

Conversely, assume that $G(f)$ is not strongly-closed. Then there exists a point $(p, q) \notin G(f)$ such that for every open $U \subset X$ and $V \subset Y$ containing $p$ and $q$, respectively, $(U \times \text{cl} (V)) \cap G(f) \neq \emptyset$. Define $F_1 = \{U_\alpha: U_\alpha \text{ is an open set containing } p \text{ and } \alpha \in \Delta_1\}$, $F_2 = \{\text{cl} (V_\beta): V_\beta \text{ is an open set containing } q \text{ and } \beta \in \Delta_2\}$, and $F = \{f(A_{(\alpha, \beta)}): (\alpha, \beta) \in \Delta_1 \times \Delta_2\}$ where $f(A_{(\alpha, \beta)}) = \{x \in U_\alpha: (x, f(x)) \in A_{(\alpha, \beta)}\}$. Then $F$ is a filterbase in $X$ with the property that $F$ converges to $p$, $f(F)$ $r$-converges to $q$, and $f(p) \neq q$. This completes the proof.

**Corollary 1.2.** Let $f: X \to Y$ be given. Then $G(f)$ is strongly-closed if and only if for each net $\{x_\alpha\}$ in $X$ such that $\{x_\alpha\}$ converges to some $p$ in $X$ and $\{f(x_\alpha)\}$ $r$-converges to some $q$ in $Y$, $f(p) = q$.

We remark that by using Corollary 1.2 it is not difficult to show that every continuous mapping from a space $X$ into a Hausdorff space $Y$ has a strongly-closed graph. Of course, the converse is not true.

Let $\{E_\alpha\}$ be a net of sets in a topological space $X$. A point $p$ in $X$ belongs to the lim sup $E_\alpha$ (lim inf $E_\alpha$) if and only if $\{E_\alpha\}$ frequently (resp., eventually) intersects each open set containing $p$ [3, p. 495].

**Theorem 1.3.** Let $f: X \to Y$ be a map. Then the following are equivalent:

(a) The function $f$ has a strongly-closed graph.

(b) If $\{y_\alpha\}$ is a net in $Y$ such that $\{y_\alpha\} \to q \in Y$, then $\lim \sup (f^{-1}(y_\alpha)) \subset f^{-1}(q)$.

(c) If $\{y_\alpha\}$ is a net in $Y$ such that $\{y_\alpha\} \to q \in Y$, then $\lim \inf (f^{-1}(y_\alpha)) \subset f^{-1}(q)$.

**Proof.** The proof parallels that of Theorem 1.2 of [3, p. 496] and is therefore omitted.

We remark that if $\{x_\alpha\}$ is a net in

$$
\prod_\beta \{Y_\beta: \beta \in \Delta\}
$$

(where $x_\alpha = \{y^0_\beta\}$), then $\{x_\alpha\} \to \{y^0_\beta\}$ if and only if $y^0_\beta \to y^0_\beta$ for each $\beta \in \Delta$. Using this observation along with Corollary 1.2, it is not difficult to show that the following two product theorems for strongly-closed graphs hold.

**Theorem 1.4.** Let $f_\beta: X \to Y_\beta$ be a map for each $\beta \in \Delta$ and define a map $f: X \to \prod_\beta Y_\beta$ by $x \to \{f_\beta(x)\}$. Then $G(f)$ is strongly-closed if and only if each $G(f_\beta)$ is strongly-closed.

**Theorem 1.5.** Let $f: \prod_\beta X_\beta \to \prod_\beta Y_\beta$, $\beta \in \Delta$, be defined by $\{x_\beta\}$
→ \{f_\beta(x_\beta)\}

where each \(f_\beta: X_\beta \to Y_\beta\) is a map. Then \(G(f)\) is strongly-closed if

and only if each \(G(f_\beta)\) is strongly-closed.

A space \(X\) is said to be \(H(i)\) if every open filterbase on \(X\) has nonvoid adherence [9, p. 465]. It is noted in [10, p. 132] that a space \(X\) is \(H(i)\) (where \(X\) is assumed to have no separation properties unless otherwise denoted) if and only if every open cover of \(X\) has a finite subcollection whose closures cover \(X\). We define a subset \(K \subset X\) to be an \(H(i)\) subset of \(X\) if and only if every open covering of \(X\) has a finite subcollection whose closures cover \(X\).

We next give some characterizations of \(H(i)\) subsets.

**Theorem 1.6.** Let \(X\) be a topological space and let \(A\) be a subset of \(X\). Then the following are equivalent:

(a) \(A\) is an \(H(i)\) subset.

(b) Each filterbase \(F = \{A_\alpha : \alpha \in \Delta\}\) in \(A\) \(-\)-accumulates to some point \(a\) in \(A\).

(c) Each maximal filterbase \(M = \{M_\beta : \beta \in \Delta\}\) in \(A\) \(-\)-converges to some point \(a\) in \(A\).

(d) Each net \(\tau: D \to A\) \(-\)-accumulates to some point \(a\) in \(A\).

(e) Each universal net \(\rho: D \to A\) \(-\)-converges to some point \(a\) in \(A\).

**Proof.** The proof is similar to that of Theorem 2 and 3 of [4] if we keep the subset \(A\) of \(X\) fixed.

**Theorem 1.7.** Let \(f: X \to Y\) be a mapping with a strongly-closed graph. Then \(f^{-1}(A)\) is closed in \(X\) for each \(H(i)\) subset \(A \subset Y\).

**Proof.** Let \(A\) be an \(H(i)\) subset of \(Y\) and suppose that \(f^{-1}(A)\) is not closed in \(X\). Then there exists a filterbase \(F\) in \(f^{-1}(A)\) converging to some point \(p \in cl(f^{-1}(A))\) (where \(p \notin f^{-1}(A)\)). Now there exists a maximal filterbase \(M\) in \(f^{-1}(A)\) stronger than \(F\) which also converges to \(p\). Then \(f(M)\) is a maximal filterbase in \(A\) and by Theorem 1.6(c), \(f(M)\) \(-\)-converges to some \(q \in A\). Theorem 1.1 then gives \(f(p) = q\). Therefore, \(p \in f^{-1}(A)\) which is a contradiction.

Theorem 2.3 of [3, p. 499] shows that if \(f: X \to Y\) has a closed graph, then \(f^{-1}(K)\) is closed for each compact \(K \subset Y\). However, if \(f: X \to Y\) has a closed graph and if \(K\) is an \(H(i)\) subset of \(Y\), then \(f^{-1}(K)\) need not be closed in \(X\). To see this, let \(Y = [0, 1] \times [0, 1]\) have as a subbase the usual open sets in \(Y\) along with the collection \(\{Y \setminus (B \times \{0\}) : B \subset [0, 1]\}\). We note that \(Y\) is an \(H\)-closed Urysohn space. Now, let \(X = [0, 1]\) have the usual topology and define \(f: X \to Y\) by \(f(x) = (x, 0)\) if \(x \neq 0\) and \(f(0) = (1, 1)\). It is not difficult to see that \(f\) has a closed graph but \(G(f)\) is not strongly-closed. Define \(K = \{(x, y) \in Y : 0 \leq y \leq \frac{1}{2}\}\). It follows that \(K\) is an \(H(i)\) subset in \(Y\) and \(f^{-1}(K) = (0, 1]\) is not closed in \(X\).

2. \(R\)-subcontinuous mappings. A map \(f: X \to Y\) is \(r\)-subcontinuous with respect to \(A \subset Y\) if and only if each convergent net \(\{x_\alpha\}\) in \(X\) has a subnet \(\{x_{\alpha\beta}\}\) such that \(\{f(x_{\alpha\beta})\}\) \(-\)-converges in \(Y\) to some point \(a \in A\). If \(A = Y\) we
say that \( f : X \to Y \) is \( r \)-subcontinuous. Thus, we see that an \( r \)-subcontinuous function is a generalization of a weakly-continuous function. Also, in view of Theorem 1.6(d) and the fact that a net \( \emptyset \) in \( X \) \( r \)-accumulates at a point \( p \in X \) if and only if \( \emptyset \) has a subnet \( r \)-converging to \( p \), we see that the concept of an \( r \)-subcontinuous function is a generalization of a mapping whose range is an \( H(i) \) subset.

The function defined in the example following Theorem 1.7 is an example of a function which is \( r \)-subcontinuous but not weakly-continuous. As another example of an \( r \)-subcontinuous function, let \( Y \) be the space defined in the example following Theorem 1.7 and let \( X = \{ 0 \} \cup \{ 1/n : n \in N \} \) have the usual subspace topology of the reals. Define \( f : X \to Y \) by \( f(1) = (1,0), f(1/2n) = (1/2n,0), f(1/(2n + 1)) = (1,1) \), and \( f(0) = (1,1) \). Then it follows that \( f \) is \( r \)-subcontinuous but not weakly-continuous.

**Theorem 2.1.** Let \( f : X \to Y \) be a map. Then \( f \) maps compact sets \( K \) onto \( H(i) \) subsets \( f(K) \subset Y \) if and only if \( f|K : K \to Y \) is \( r \)-subcontinuous with respect to \( f(K) \) for each compact \( K \subset X \).

**Proof.** The proof follows almost directly from Theorem 1.6(d) and the definition of an \( r \)-subcontinuous function. (The result also parallels that of Theorem 3.1(b) of [3, p. 498].)

**Theorem 2.2.** Let \( f : X \to Y \) be a map with a strongly-closed graph. Then \( f \) is weakly-continuous if and only if \( f \) is \( r \)-subcontinuous.

**Proof.** Only the sufficiency requires proof. Suppose that \( \{ x_\alpha \} \) is a net in \( X \) such that \( \{ x_\alpha \} \) converges to \( p \) but \( \{ f(x_\alpha) \} \) does not \( r \)-converge to \( f(p) \). Then \( \{ f(x_\alpha) \} \) has a subnet, \( \{ f(x_{\alpha_\beta}) \} \), with the property that no subnet \( r \)-converges to \( f(p) \). Since \( f \) is \( r \)-subcontinuous, there exists a subnet, \( \{ x_\gamma \} \), of the net \( \{ x_{\alpha_\beta} \} \) such that \( \{ f(x_\gamma) \} \) \( r \)-converges to some \( q \) in \( Y \). The strongly-closed graph condition gives \( f(p) = q \) which is a contradiction. We conclude that \( f \) is weakly-continuous.

**Theorem 2.3.** Let \( f : X \to Y \) be an open function with a closed graph from a space \( X \) into a semiregular space \( Y \). If \( f \) is \( r \)-subcontinuous, then \( f \) is continuous.

**Proof.** Let \( (x,y) \notin G(f) \). By the Lemma of [8, p. 931] there exist open sets \( U \subset X \) and \( V \subset Y \) containing \( x \) and \( y \), respectively, such that \( f(U) \cap V = \emptyset \). Since \( f \) is open, \( f(U) \cap \text{cl}(V) = \emptyset \) which implies that \( f \) has a strongly-closed graph. Therefore, by Theorem 2.2, \( f \) is weakly-continuous. Now let \( V \) be a regular-open set containing \( f(x) \). Since \( f \) is weakly-continuous, there exists an open \( U \) containing \( x \) such that \( f(U) \subset \text{cl}(V) \). Therefore, \( f(U) \subset \text{Int}(\text{cl}(f(U))) \subset \text{Int}(\text{cl}(V)) = V \) which shows that \( f \) is continuous.

Theorem 1 of [6] and Corollary 2 of [9] give necessary and sufficient conditions for a \( T_i \) space \( Y \) to be compact. Using our knowledge of \( r \)-subcontinuous functions and \( H(i) \) spaces, we next give a necessary and sufficient condition for a \( T_i \) space \( Y \) to be \( H(i) \). In so doing, we rely on a particular class of topological space defined in [6] called class \( S \).

**Theorem 2.4.** A \( T_i \)-space \( Y \) is \( H(i) \) if and only if for every topological space \( X \) belonging to \( S \), each mapping \( f : X \to Y \) with a closed graph is \( r \)-subcontinuous.
Proof. In view of the remark at the beginning of §2, it is clear that only the sufficiency part of the theorem requires proof. Suppose that $Y$ is not $H(i)$. Then by Theorem 1.6(d), there exists a net $f: D \to Y$ which has no $r$-accumulation point. Let $\infty \not\in D$ and define $X = D \cup \{\infty\}$. It follows that the power set of $D$, $P(D)$, together with the collection \( \{T_d \cup \{\infty\}: d \in D\} \) is a base for a completely normal and fully normal Hausdorff space (see Theorem 1 of [6]). Let $p \in Y$ and define $g: X \to Y$ by $g|D = f$ and $g(\infty) = p$. It follows (as in the proof of Theorem 1 of [6]) that $G(g)$ is closed in $X \times Y$. Now, the identity map $I: D \to D \subseteq X$ defines a net in $X$ with the property that the net $I$ converges to $\infty$. However, if $y$ is any point in $Y$, then there exists an open set $W(y)$ containing $y$ and some $d_y \in D$ such that $g(T_{d_y}) \cap \text{cl} (W(y)) = \emptyset$. Consequently, $g$ is not $r$-subcontinuous at $x = \infty$.

3. Feebly compact and minimal Hausdorff spaces. We say that a space $X$ is feebly compact if and only if every countable open filterbase has an adherent point [1, p. 107]. It is noted in [9, p. 469] that a space $X$ is feebly compact if and only if every countable open covering of $X$ possesses a finite subcollection whose closures cover $X$. Theorem 11 of [9] gives a condition when a space $Y$ is feebly compact. We next give a sufficient condition for a space $Y$ to be feebly compact.

**Theorem 3.1.** A space $Y$ is feebly compact if every sequence in $Y$ has an $r$-accumulation point.

**Proof.** Assume that $Y$ is not feebly compact. Then there is a countable open covering \( \{U_i: i \in \mathbb{N}\} \) that has no finite subcollection whose closures cover $Y$. Thus, for each $n \in \mathbb{N}$, we can find some $y_n \in Y$ such that $y_n \in Y - \bigcup_{i=1}^{n} \text{cl} (U_i)$. Therefore, the map $f: N \to Y$ (defined by $f(n) = y_n$) defines a sequence in $Y$ which has no $r$-accumulation point. We conclude that $Y$ is feebly compact.

We note that in a regular space $Y$, a net converges (accumulates) to a point $p \in Y$ if and only if it $r$-converges (resp., $r$-accumulates) to $p$. Using this observation we next show that the converse of Theorem 3.1 need not hold.

**Example 3.2.** Let $Y = [0, \Omega] \times [0, \omega] - \{(\Omega, \omega)\}$ be the space defined in Example 2 of [2, p. 231]. Example 2 of [2, p. 231] shows that $Y$ is a completely regular pseudocompact Hausdorff space that is not countably compact. By Theorem 3.8 of [2, p. 232], $Y$ is feebly compact. But since $Y$ is not countably compact, there exists a sequence in $Y$ which has no $r$-accumulation point.

We say that a point $p \in X$ is an $r$-cluster point of $K \subseteq X$ if for every open set $U \subseteq X$ containing $p$, $\text{cl} (U) \cap (K - \{p\}) \neq \emptyset$. We note that in a Hausdorff space $X$, a point $p \in X$ is an $r$-cluster point of $K \subseteq X$ if and only if the closure of each open set $U$ containing $p$ contains infinitely many points of $K$.

**Lemma 3.3.** In a topological space $Y$ the following are equivalent:

(a) Every countably infinite subset of $Y$ has at least one $r$-cluster point.

(b) Every sequence in $Y$ has an $r$-accumulation point.

**Proof.** The straightforward proof is omitted.

**Theorem 3.4.** A space $Y$ is feebly compact if every countably infinite subset of $Y$ has at least one $r$-cluster point.
PROOF. The result follows from Theorem 3.1 and Lemma 3.3.

A space \((X, T)\) is called first countable and minimal Hausdorff if \(T\) is first countable and Hausdorff, and if no first countable topology on \(X\) which is strictly weaker than \(T\) is Hausdorff. Theorem 6.2 of [1, p. 107] shows that a first countable Hausdorff space \(X\) is first countable and minimal Hausdorff if every countable filterbase on \(X\) with a unique adherent point is convergent. We show (after Lemma 3.5) that a first countable Hausdorff \(X\) is first countable and minimal Hausdorff if each sequence in \(X\) with a unique \(r\)-accumulation point is convergent.

**Lemma 3.5.** Suppose that \(X\) is a Hausdorff space with the property that every sequence in \(X\) with a unique \(r\)-accumulation point is convergent. Then every sequence in \(X\) has an \(r\)-accumulation point.

**Proof.** Suppose there is a sequence, \((x_n)\), in \(X\) which has no \(r\)-accumulation point. Fix \(p \in X\) and define a sequence, \((z_n)\), by \(z_n = p\) if \(n\) is odd and \(z_n = x_{n/2}\) if \(n\) is even. It is clear that \(p\) is the unique \(r\)-accumulation point of \((z_n)\) and that \((z_n)\) does not converge to \(p\).

**Theorem 3.6.** Let \((X, T)\) be a first countable Hausdorff space. Then \(X\) is first countable and minimal Hausdorff if every sequence in \(X\) with a unique \(r\)-accumulation point is convergent.

**Proof.** Suppose that \(f: (X, T) \to (Y, \sigma)\) is a bijective continuous mapping onto a first countable Hausdorff space \(Y\). We only need to show that \(f\) is continuous. Let \((y_n)\) be a sequence in \(Y\) converging to \(y \in Y\). The sequence \((f^{-1}(y_n))\) has a unique \(r\)-accumulation point because if \(p \in X\) is an \(r\)-accumulation point of the sequence \((f^{-1}(y_n))\), then the continuity of \(f\) shows that \(f(p)\) is an \(r\)-accumulation point of \((y_n)\) and, therefore, since \(Y\) is Hausdorff, \(f(p) = y\). Thus, \(f^{-1}(y)\) is the unique \(r\)-accumulation point of \((f^{-1}(y_n))\). This implies, by hypothesis, that \((f^{-1}(y_n))\) converges to \(f^{-1}(y)\) showing that \(f^{-1}\) is continuous. We conclude that \((X, T)\) is minimal Hausdorff.

In view of Theorem 3.1, we can give an additional sufficient condition for a space \(X\) to be feebly compact. We rely on the space \(\bar{N}\) defined in [6, p. 524].

**Theorem 3.7.** A Hausdorff space \(Y\) is feebly compact if each mapping of \(\bar{N}\) into \(Y\) with a strongly-closed graph is weakly-continuous.

**Proof.** Suppose that \(Y\) is not feebly compact. Then by Theorem 3.1, there exists a sequence, \(f: N \to Y\), in \(Y\) which has no \(r\)-accumulation point. The proof now parallels that of Theorem 2.4.

**Corollary 3.8.** Let \(Y\) be a first countable semiregular Hausdorff space. Then \(Y\) is first countable and minimal Hausdorff if each mapping of \(\bar{N}\) into \(Y\) with a strongly-closed graph is weakly-continuous.

**Proof.** The result follows from Theorem 3.7 and Theorem 6.2(a) (part (ii)) of [1, p. 107].

**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS AT PINE BLUFF, PINE BLUFF, ARKANSAS 71601