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REMARKS ON $H(i)$ SPACES AND STRONGLY-CLOSED GRAPHS

LARRY L. HERRINGTON

ABSTRACT. This paper gives characterizations of $H(i)$ spaces and strongly-closed graphs.

If X and Y topological spaces, we say a mapping $f: X \to Y$ has a strongly-closed graph if for each $(x,y) \notin G(f) = \{(x,f(x)): x \in X\} \subset X \times Y$ there exist open sets U and V containing x and y, respectively, such that $(U \times \mathrm{cl}\,(V)) \cap G(f) = \varnothing$ (where $\mathrm{cl}\,(V)$ denotes the closure of V). We say that a filterbase $F = \{A_\alpha \subset A: \alpha \in \Delta\}$ in $A \subset X$ r-converges[1] to a point a in A ($F \to_r a$) if for every open $U \subset X$ containing a there exists an A_α in F such that $A_\alpha \subset \mathrm{cl}\,(U)$. The filterbase F r-accumulates to a point $a \in A$ ($F \propto_r a$) if for every A_α in F and for every open $U \subset X$ containing a, $A_\alpha \cap \mathrm{cl}\,(U) \neq \varnothing$. In a like manner we define the concept of a net $\mathcal{O}: D \to A \subset X$ r-converging to $a \in A$ ($\mathcal{O} \to_r a$) and r-accumulating to a point $a \in A$ ($\mathcal{O} \propto_r a$). Of course, if $\mathcal{O}: D \to A \subset X$ is a net in A, then the family $F(\mathcal{O}) = \{\mathcal{O}(T_b): b \in D\}$ (where $T_b = \{a \in D: a \geqslant b\}$) is a filterbase in $A \subset X$ and it is routine to verify that (a) $F(\mathcal{O}) \to_r a \in A$ if and only if $\mathcal{O} \to_r a$ and (b) $F(\mathcal{O}) \propto_r a \in A$ if and only if $\mathcal{O} \propto_r a$. A mapping $f: X \to Y$ is weakly-continuous at $x \in X$ if for each open V containing $f(x)$ there exists an open U containing x such that $f(U) \subset \mathrm{cl}\,(V)$ [7]. It is noted in [5] that a mapping $f: X \to Y$ is weakly-continuous at $x \in X$ if and only if for each net $\{x_\alpha\}$ in X converging to x, the net $\{f(x_\alpha)\} \to_r f(x)$. A subset $A \subset X$ is called regular-open (regular-closed) if it is the interior of its closure (resp., closure of its interior) [2, p. 92].

Characterizations of C-compact, H-closed, and minimal Hausdorff spaces are given in [4] and [5] by using the concepts of r-convergence, r-accumulation, strongly-closed graphs, and weakly-continuous functions. In this paper we further investigate strongly-closed graphs and give in addition some characterizations of feebly compact and $H(i)$ spaces.

1. **Strongly-closed graphs and $H(i)$ subsets.** We first characterize strongly-closed graphs in terms of filterbases.

THEOREM 1.1. *Let $f: X \to Y$ be given. Then $G(f)$ is strongly-closed if and only*

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[1] The concept of r-convergence and r-accumulation was first introduced by N. V. Veličko under the names of θ-convergence and θ-contact point respectively.

if for each filterbase F in X such that F converges to some p in X and $f(F)$ r-converges to some q in Y, $f(p) = q$.

PROOF. Suppose that $G(f)$ is strongly-closed and let $F = \{A_\alpha : \alpha \in \Delta\}$ be a filterbase in X such that F converges to p and $f(F)$ r-converges to q. If $f(p) \neq q$, then $(p,q) \notin G(f)$. Thus there exist open sets $U \subset X$ and $V \subset Y$ containing p and q, respectively, such that $(U \times \mathrm{cl}\,(V)) \cap G(f) = \varnothing$. Since F converges to p and $f(F)$ r-converges to q, there exists an $A_\alpha \in F$ such that $A_\alpha \subset U$ and $f(A_\alpha) \subset \mathrm{cl}\,(V)$. Consequently, $(U \times \mathrm{cl}\,(V)) \cap G(f) \neq \varnothing$ which is a contradiction.

Conversely, assume that $G(f)$ is not strongly-closed. Then there exists a point $(p,q) \notin G(f)$ such that for every open $U \subset X$ and $V \subset Y$ containing p and q, respectively, $(U \times \mathrm{cl}\,(V)) \cap G(f) \neq \varnothing$. Define $F_1 = \{U_\alpha : U_\alpha$ is an open set containing p and $\alpha \in \Delta_1\}$, $F_2 = \{\mathrm{cl}\,(V_\beta) : V_\beta$ is an open set containing q and $\beta \in \Delta_2\}$, $F_3 = \{A_{(\alpha,\beta)} : A_{(\alpha,\beta)} = (U_\alpha \times \mathrm{cl}\,(V_\beta)) \cap G(f), (\alpha,\beta) \in \Delta_1 \times \Delta_2\}$, and $F = \{F_{(\alpha,\beta)} : (\alpha,\beta) \in \Delta_1 \times \Delta_2\}$ where $F_{(\alpha,\beta)} = \{x \in U_\alpha : (x, f(x)) \in A_{(\alpha,\beta)}\}$. Then F is a filterbase in X with the property that F converges to p, $f(F)$ r-converges to q, and $f(p) \neq q$. This completes the proof.

COROLLARY 1.2. *Let $f: X \to Y$ be given. Then $G(f)$ is strongly-closed if and only if for each net $\{x_\alpha\}$ in X such that $\{x_\alpha\}$ converges to some p in X and $\{f(x_\alpha)\}$ r-converges to some q in Y, $f(p) = q$.*

We remark that by using Corollary 1.2 it is not difficult to show that every continuous mapping from a space X into a Hausdorff space Y has a strongly-closed graph. Of course, the converse is not true.

Let $\{E_\alpha\}$ be a net of sets in a topological space X. A point p in X belongs to the lim sup E_α (lim inf E_α) if and only if $\{E_\alpha\}$ frequently (resp., eventually) intersects each open set containing p [3, p. 495].

THEOREM 1.3. *Let $f: X \to Y$ be a map. Then the following are equivalent*:

(a) *The function f has a strongly-closed graph.*

(b) *If $\{y_\alpha\}$ is a net in Y such that $\{y_\alpha\} \to_r q \in Y$, then* $\limsup(f^{-1}(y_\alpha)) \subset f^{-1}(q)$.

(c) *If $\{y_\alpha\}$ is a net in Y such that $\{y_\alpha\} \to_r q \in Y$, then* $\liminf(f^{-1}(y_\alpha)) \subset f^{-1}(q)$.

PROOF. The proof parallels that of Theorem 1.2 of [3, p. 496] and is therefore omitted.

We remark that if $\{x_\alpha\}$ is a net in

$$\prod_\beta \{Y_\beta : \beta \in \Delta\}$$

(where $x_\alpha = \{y_\beta^\alpha\}$), then $\{x_\alpha\} \to_r \{y_\beta^0\}$ if and only if $y_\beta^\alpha \to_r y_\beta^0$ for each $\beta \in \Delta$. Using this observation along with Corollary 1.2, it is not difficult to show that the following two product theorems for strongly-closed graphs hold.

THEOREM 1.4. *Let $f_\beta: X \to Y_\beta$ be a map for each $\beta \in \Delta$ and define a map $f: X \to \prod_\beta Y_\beta$ by $x \to \{f_\beta(x)\}$. Then $G(f)$ is strongly-closed if and only if each $G(f_\beta)$ is strongly-closed.*

THEOREM 1.5. *Let $f: \prod_\beta X_\beta \to \prod_\beta Y_\beta$, $\beta \in \Delta$, be defined by $\{x_\beta\}$*

$\to \{f_\beta(x_\beta)\}$ where each $f_\beta: X_\beta \to Y_\beta$ is a map. Then $G(f)$ is strongly-closed if and only if each $G(f_\beta)$ is strongly-closed.

A space X is said to be $H(i)$ if every open filterbase on X has nonvoid adherence [9, p. 465]. It is noted in [10, p. 132] that a space X is $H(i)$ (where X is assumed to have no separation properties unless otherwise denoted) if and only if every open covering of X has a finite subcollection whose closures cover X. We define a subset $K \subset X$ to be an $H(i)$ subset of X if for every open cover $\{U_\alpha: U_\alpha$ is open in X and $\alpha \in \Delta\}$ of K, there exists a finite subcollection $\{U_{\alpha_i}: i = 1, 2, \ldots, n\}$ such that $K \subset \bigcup_{i=1}^n \operatorname{cl}(U_{\alpha_i})$. It follows that a subset $K \subset X$ is an $H(i)$ subset if and only if every open filterbase $F = \{O_\alpha: O_\alpha$ is open in X, $O_\alpha \cap K \neq \varnothing$, and $\alpha \in \Delta\}$ in X has at least one accumulation point $k \in K$.

We next give some characterizations of $H(i)$ subsets.

THEOREM 1.6. *Let X be a topological space and let A be a subset of X. Then the following are equivalent*:

(a) *A is an $H(i)$ subset.*

(b) *Each filterbase $F = \{A_\alpha: \alpha \in \Delta\}$ in A r-accumulates to some point a in A.*

(c) *Each maximal filterbase $M = \{M_\beta: \beta \in \Delta\}$ in A r-converges to some point a in A.*

(d) *Each net $\mathcal{O}: D \to A$ r-accumulates to some point a in A.*

(e) *Each universal net $\mathcal{O}: D \to A$ r-converges to some point a in A.*

PROOF. The proof is similar to that of Theorem 2 and 3 of [4] if we keep the subset A of X fixed.

THEOREM 1.7. *Let $f: X \to Y$ be a mapping with a strongly-closed graph. Then $f^{-1}(A)$ is closed in X for each $H(i)$ subset $A \subset Y$.*

PROOF. Let A be an $H(i)$ subset of Y and suppose that $f^{-1}(A)$ is not closed in X. Then there exists a filterbase F in $f^{-1}(A)$ converging to some point $p \in \operatorname{cl}(f^{-1}(A))$ (where $p \notin f^{-1}(A)$). Now there exists a maximal filterbase M in $f^{-1}(A)$ stronger than F which also converges to p. Then $f(M)$ is a maximal filterbase in A and by Theorem 1.6(c), $f(M)$ r-converges to some $q \in A$. Theorem 1.1 then gives $f(p) = q$. Therefore, $p \in f^{-1}(A)$ which is a contradiction.

Theorem 3.6 of [3, p. 499] shows that if $f: X \to Y$ has a closed graph, then $f^{-1}(K)$ is closed for each compact $K \subset Y$. However, if $f: X \to Y$ has a closed graph and if K is an $H(i)$ subset of Y, then $f^{-1}(K)$ need not be closed in X. To see this, let $Y = [0,1] \times [0,1]$ have as a subbase the usual open sets in Y along with the collection $\{Y - (B \times \{0\}): B \subset [0,1]\}$. We note that Y is an H-closed Urysohn space. Now, let $X = [0,1]$ have the usual topology and define $f: X \to Y$ by $f(x) = (x, 0)$ if $x \neq 0$ and $f(0) = (1,1)$. It is not difficult to see that f has a closed graph but $G(f)$ is not strongly-closed. Define $K = \{(x,y) \in Y: 0 \leqslant y \leqslant \frac{1}{2}\}$. It follows that K is an $H(i)$ subset in Y and $f^{-1}(K) = (0,1]$ is not closed in X.

2. ***R*-subcontinuous mappings.** A map $f: X \to Y$ is r-subcontinuous with respect to $A \subset Y$ if and only if each convergent net $\{x_\alpha\}$ in X has a subnet $\{x_{\alpha_\beta}\}$ such that $\{f(x_{\alpha_\beta})\}$ r-converges in Y to some point $a \in A$. If $A = Y$ we

say that $f: X \to Y$ is r-subcontinuous. Thus, we see that an r-subcontinuous function is a generalization of a weakly-continuous function. Also, in view of Theorem 1.6(d) and the fact that a net $\mathcal{O}$ in X r-accumulates at a point $p \in X$ if and only if $\mathcal{O}$ has a subnet r-converging to p, we see that the concept of an r-subcontinuous function is a generalization of a mapping whose range is an $H(i)$ subset.

The function defined in the example following Theorem 1.7 is an example of a function which is r-subcontinuous but not weakly-continuous. As another example of an r-subcontinuous function, let Y be the space defined in the example following Theorem 1.7 and let $X = \{0\} \cup \{1/n: n \in N\}$ have the usual subspace topology of the reals. Define $f: X \to Y$ by $f(1) = (1,0)$, $f(1/2n) = (1/2n, 0)$, $f(1/(2n+1)) = (1,1)$, and $f(0) = (1,1)$. Then it follows that f is r-subcontinuous but not weakly-continuous.

THEOREM 2.1. *Let $f: X \to Y$ be a map. Then f maps compact sets K onto $H(i)$ subsets $f(K) \subset Y$ if and only if $f|K: K \to Y$ is r-subcontinuous with respect to $f(K)$ for each compact $K \subset X$.*

PROOF. The proof follows almost directly from Theorem 1.6(d) and the definition of an r-subcontinuous function. (The result also parallels that of Theorem 3.1(b) of [3, p. 498].)

THEOREM 2.2. *Let $f: X \to Y$ be a map with a strongly-closed graph. Then f is weakly-continuous if and only if f is r-subcontinuous.*

PROOF. Only the sufficiency requires proof. Suppose that $\{x_\alpha\}$ is a net in X such that $\{x_\alpha\}$ converges to p but $\{f(x_\alpha)\}$ does not r-converge to $f(p)$. Then $\{f(x_\alpha)\}$ has a subnet, $\{f(x_{\alpha_\beta})\}$, with the property that no subnet r-converges to $f(p)$. Since f is r-subcontinuous, there exists a subnet, $\{x_\gamma\}$, of the net $\{x_{\alpha_\beta}\}$ such that $\{f(x_\gamma)\}$ r-converges to some q in Y. The strongly-closed graph condition gives $f(p) = q$ which is a contradiction. We conclude that f is weakly-continuous.

THEOREM 2.3. *Let $f: X \to Y$ be an open function with a closed graph from a space X into a semiregular space Y. If f is r-subcontinuous, then f is continuous.*

PROOF. Let $(x,y) \notin G(f)$. By the Lemma of [8, p. 931] there exist open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap V = \varnothing$. Since f is open, $f(U) \cap \mathrm{cl}\,(V) = \varnothing$ which implies that f has a strongly-closed graph. Therefore, by Theorem 2.2, f is weakly-continuous. Now let V be a regular-open set containing $f(x)$. Since f is weakly-continuous, there exists an open U containing x such that $f(U) \subset \mathrm{cl}\,(V)$. Therefore, $f(U) \subset \mathrm{Int(cl)(f(U))} \subset \mathrm{Int(cl)(V)} = V$ which shows that f is continuous.

Theorem 1 of [6] and Corollary 2 of [9] give necessary and sufficient conditions for a T_1 space Y to be compact. Using our knowledge of r-subcontinuous functions and $H(i)$ spaces, we next give a necessary and sufficient condition for a T_1-space Y to be $H(i)$. In so doing, we rely on a particular class of topological space defined in [6] called class S.

THEOREM 2.4. *A T_1-space Y is $H(i)$ if and only if for every topological space X belonging to S, each mapping $f: X \to Y$ with a closed graph is r-subcontinuous.*

PROOF. In view of the remark at the beginning of §2, it is clear that only the sufficiency part of the theorem requires proof. Suppose that Y is not $H(i)$. Then by Theorem 1.6(d), there exists a net $f: D \to Y$ which has no r-accumulation point. Let $\infty \notin D$ and define $X = D \cup \{\infty\}$. It follows that the power set of D, $P(D)$, together with the collection $\{T_d \cup \{\infty\}: d \in D\}$ is a base for a completely normal and fully normal Hausdorff space (see Theorem 1 of [6]). Let $p \in Y$ and define $g: X \to Y$ by $g|D = f$ and $g(\infty) = p$. It follows (as in the proof of Theorem 1 of [6]) that $G(g)$ is closed in $X \times Y$. Now, the identity map $1: D \to D \subset X$ defines a net in X with the property that the net 1 converges to ∞. However, if y is any point in Y, then there exists an open set $W(y)$ containing y and some $d_y \in D$ such that $g(T_{d_y}) \cap \mathrm{cl}\,(W(y)) = \varnothing$. Consequently, g is not r-subcontinuous at $x = \infty$.

3. **Feebly compact and minimal Hausdorff spaces.** We say that a space X is feebly compact if and only if every countable open filterbase has an adherent point [1, p. 107]. It is noted in [9, p. 469] that a space X is feebly compact if and only if every countable open covering of X possesses a finite subcollection whose closures cover X. Theorem 11 of [9] gives a condition when a space Y is feebly compact. We next give a sufficient condition for a space Y to be feebly compact.

THEOREM 3.1. *A space Y is feebly compact if every sequence in Y has an r-accumulation point.*

PROOF. Assume that Y is not feebly compact. Then there is a countable open covering $\{U_i: i \in N\}$ that has no finite subcollection whose closures cover Y. Thus, for each $n \in N$, we can find some $y_n \in Y$ such that $y_n \in Y - \bigcup_{i=1}^{n} \mathrm{cl}\,(U_i)$. Therefore, the map $f: N \to Y$ (defined by $f(n) = y_n$) defines a sequence in Y which has no r-accumulation point. We conclude that Y is feebly compact.

We note that in a regular space Y, a net converges (accumulates) to a point $p \in Y$ if and only if it r-converges (resp., r-accumulates) to p. Using this observation we next show that the converse of Theorem 3.1 need not hold.

EXAMPLE 3.2. Let $Y = [0, \Omega] \times [0, \omega] - \{(\Omega, \omega)\}$ be the space defined in Example 2 of [2, p. 231]. Example 2 of [2, p. 231] shows that Y is a completely regular pseudocompact Hausdorff space that is not countably compact. By Theorem 3.8 of [2, p. 232], Y is feebly compact. But since Y is not countably compact, there exists a sequence in Y which has no r-accumulation point.

We say that a point $p \in X$ is an r-cluster point of $K \subset X$ if for every open set $U \subset X$ containing p, $\mathrm{cl}\,(U) \cap (K - \{p\}) \neq \varnothing$. We note that in a Hausdorff space X, a point $p \in X$ is an r-cluster point of $K \subset X$ if and only if the closure of each open set U containing p contains infinitely many points of K.

LEMMA 3.3. *In a topological space Y the following are equivalent:*
(a) *Every countably infinite subset of Y has at least one r-cluster point.*
(b) *Every sequence in Y has an r-accumulation point.*

PROOF. The straightforward proof is omitted.

THEOREM 3.4. *A space Y is feebly compact if every countably infinite subset of Y has at least one r-cluster point.*

PROOF. The result follows from Theorem 3.1 and Lemma 3.3.

A space (X, T) is called first countable and minimal Hausdorff if T is first countable and Hausdorff, and if no first countable topology on X which is strictly weaker than T is Hausdorff. Theorem 6.2 of [1, p. 107] shows that a first countable Hausdorff space X is first countable and minimal Hausdorff if every countable filterbase on X with a unique adherent point is convergent. We show (after Lemma 3.5) that a first countable Hausdorff X is first countable and minimal Hausdorff if each sequence in X with a unique r-accumulation point is convergent.

LEMMA 3.5. *Suppose that X is a Hausdorff space with the property that every sequence in X with a unique r-accumulation point is convergent. Then every sequence in X has an r-accumulation point.*

PROOF. Suppose there is a sequence, (x_n), in X which has no r-accumulation point. Fix $p \in X$ and define a sequence, (z_n), by $z_n = p$ if n is odd and $z_n = x_{n/2}$ if n is even. It is clear that p is the unique r-accumulation point of (z_n) and that (z_n) does not converge to p.

THEOREM 3.6. *Let (X, T) be a first countable Hausdorff space. Then X is first countable and minimal Hausdorff if every sequence in X with a unique r-accumulation point is convergent.*

PROOF. Suppose that f: $(X, T) \to (Y, \sigma)$ is a bijective continuous mapping onto a first countable Hausdorff space Y. We only need to show that f is continuous. Let (y_n) be a sequence in Y converging to $y \in Y$. The sequence $(f^{-1}(y_n))$ has a unique r-accumulation point because if $p \in X$ is an r-accumulation point of the sequence $(f^{-1}(y_n))$, then the continuity of f shows that $f(p)$ is an r-accumulation point of (y_n) and, therefore, since Y is Hausdorff, $f(p) = y$. Thus, $f^{-1}(y)$ is the unique r-accumulation point of $(f^{-1}(y_n))$. This implies, by hypothesis, that $(f^{-1}(y_n))$ converges to $f^{-1}(y)$ showing that f^{-1} is continuous. We conclude that (X, T) is minimal Hausdorff.

In view of Theorem 3.1, we can give an additional sufficient condition for a space X to be feebly compact. We rely on the space $\overline{N}$ defined in [6, p. 524].

THEOREM 3.7. *A Hausdorff space Y is feebly compact if each mapping of $\overline{N}$ into Y with a strongly-closed graph is weakly-continuous.*

PROOF. Suppose that Y is not feebly compact. Then by Theorem 3.1, there exists a sequence, f: $N \to Y$, in Y which has no r-accumulation point. The proof now parallels that of Theorem 2.4.

COROLLARY 3.8. *Let Y be a first countable semiregular Hausdorff space. Then Y is first countable and minimal Hausdorff if each mapping of $\overline{N}$ into Y with a strongly-closed graph is weakly-continuous.*

PROOF. The result follows from Theorem 3.7 and Theorem 6.2(a) (part (ii)) of [1, p. 107].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS AT PINE BLUFF, PINE BLUFF, ARKANSAS 71601