A FORMULA FOR STIEFEL-WHITNEY HOMOLOGY CLASSES

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The purpose of this paper is to define for mod 2 Euler spaces a formula which enables one to compute the Stiefel-Whitney homology classes in the original triangulation without passing to the first barycentric subdivision. The formula has a somewhat tenuous connection to the Steenrod reduced squares. In the case when we are dealing with a smooth triangulation, the Wu formulae [7] and the Whitney theorem [4] establish such a connection. The authors would like to thank S. Halperin and D. Toledo for a copy of their preprint [5]; the use of their map $\phi$ (see §2) simplifies an earlier proof of the main theorem. The homology theory used is that based on infinite chains.

1. Statement of the theorem. Let $K$ be a finite-dimensional, locally finite simplicial complex. $K$ is said to be a mod 2 Euler space if the link of every simplex in $K$ has even Euler characteristic [9]. The $p$th Stiefel-Whitney class of $K$, denoted $\omega_p(K)$, is the $p$-dimensional mod 2 homology class which has a representative, the $p$-dimensional chain consisting of all $p$-simplexes in the first barycentric subdivision of $K$—this chain is a cycle for each $p$ iff $K$ is a mod 2 Euler space.

From now on we assume that $K$ is given an ordering of its vertices and any representation of a simplex in $K$ is written with its vertices in increasing order. We now recall a definition due to Steenrod [8]. Let $s$ be a $p$-simplex in $K$, say $s = \langle v_0, v_1, \ldots, v_p \rangle$. Let $t$ be another simplex which has $s$ as a face; i.e., $s \subset t$ ($s$ may be equal to $t$). Let

$B_{-1} = \text{set of vertices of } t \text{ less than } v_0,$

$B_0 = \text{set of vertices of } t \text{ strictly between } v_0 \text{ and } v_1,$

$B_m = \text{set of vertices of } t \text{ strictly between } v_m \text{ and } v_{m+1},$

$B_p = \text{set of vertices of } t \text{ greater than } v_p.$

We say that $s$ is regular in $t$, if $\#(B_m) = 0$ for every odd $m$. Let $\partial_p(t)$ denote the mod 2 chain which consists of all $p$-dimensional simplexes $s$ in $t$ so that $s$ is regular in $t$. 

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is regular in \( t \). The following is the main result in this paper.

**Theorem.** \( \sum_{\dim t \geq p} \partial_p(t) \) is a chain which represents \( \omega_p(K) \).

2. **Proof of the main theorem.** We define a simplicial map \( \phi: K' \to K \) in the following way. Pick a simplex \( s \in K \) and let \( \ell(s) \) be the smallest vertex in \( s \); map the barycenter of \( s \) to \( \ell(s) \). This defines a simplicial map \( \phi: K' \to K \) which induces the identity in homology. To establish the theorem it suffices to show that the number of \( p \)-simplexes in \( K' \) which map onto a given \( p \)-simplex \( s \) in \( K \) is congruent mod 2 to the number of simplexes \( t \), so that \( s \) is regular in \( t \). However a \( p \)-simplex \( s' \) in \( K' \) maps onto \( s \) only if \( s \) is a face of the carrier in \( K \) of \( s' \); hence the theorem is a consequence of the following

**Lemma.** Let \( \phi: K' \to K \) be defined as above and let \( s \) be a face of \( t \); then the number of \( p \)-simplexes whose carrier is \( t \) and which map onto \( s \) is

(i) odd if \( s \) is regular in \( t \),

(ii) even if \( s \) is not regular in \( t \).

**Proof.** If \( B_{-1} \) is not empty then \( s \) is not regular in \( t \) and no simplex whose carrier is \( t \) maps onto \( s \). Hence we assume that \( B_{-1} \) is empty. Now a \( p \)-simplex whose carrier is \( t \) and whose image under \( \phi \) is \( s \) must look like the following:

\[
\{(v_0, B_0, v_1, B_1, \ldots, v_p, B_p), (v_1, B'_1, v_2, B'_2, \ldots, v_p, B'_p), \ldots, (v_p, B^{(p)}_p)\}
\]

where

\[
B'_1 \subset B_1,
B''_2 \subset B'_2 \subset B_2,
\vdots
B^{(p)}_p \subset \cdots \subset B'_p \subset B_p.
\]

Denote by \( c(B_j) \), the number of ways of choosing \( j \) nondecreasing subsets of \( B_j \); hence the number of \( p \)-simplexes whose image is \( s \) is \( c(B_1) \cdot c(B_2) \cdot \ldots \cdot c(B_p) \).

**Proposition.** (1) \( c(B_j) = 1 \) if \( B_j = \emptyset \),

(2) \( c(B_j) \equiv 0 \pmod 2 \) if \( B_j \neq \emptyset \) and \( j \) is odd,

(3) \( c(B_j) \equiv 1 \pmod 2 \) if \( B_j \neq \emptyset \) and \( j \) is even.

**Proof.** Part 1 follows from the fact that \( B_j^{(k)} \) is always the null set.

Now when \( m \geq 0 \) we have that

\[
\sum_{a=0}^{m} \sum_{b=0}^{a} \binom{m}{a} \binom{a}{b} = \sum_{a=0}^{m} \binom{m}{a} \sum_{b=0}^{a} \binom{a}{b} = \binom{m}{0} \pmod 2 \equiv 1 \pmod 2.
\]

Now in general

\[
c(B_j) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \cdots \sum_{n_{j-1}=0}^{n_{j-1}} \binom{n}{n_1} \binom{n_1}{n_2} \cdots \binom{n_{j-1}}{n_j} \quad \text{where} \quad \#(B_j) = n_j.
\]

Therefore, \( \pmod 2 \) we have that
$c(B_j) = 1$ when $j$ is even.

c(B_j) = \sum_{n_1=0}^{n} \binom{n}{n_1} = 0$ when $j$ is odd since $n > 0$.

Now back to the proof of the lemma. When $s$ is regular in $t$, $B_j = \emptyset$ for $j$ odd; hence

$$c(B_1) \cdot c(B_2) \cdots \cdot c(B_p) = 1 \cdot 1 \cdots 1 \equiv 1 \pmod{2}.$$

When $s$ is not regular in $t$, some $B_j \neq \emptyset$ for $j$ odd; hence $c(B_j) = 0$ and the lemma is proven.

3. Some remarks on the formula. The above proof works verbatim in the case that the vertices are partially ordered in such a way that those of any simplex are linearly ordered, as in the barycentric subdivision ordered by dimension. It is not hard to see that in $K'$, each $p$-simplex is a regular face of an odd number of simplexes, so that

$$\sum_{\dim(t) \geq p} \partial_p(t) = \sum p\text{-simplexes of } K'.$$

It follows that the $p$-skeleta of repeated barycentric subdivisions are homologous. (E. Akin has shown that $\omega_p(K)$ is a PL invariant [1].)

When $p = 0$ it is tempting to call the coefficient of a vertex $v$ in

$$\sum_{\dim t \geq 0} \partial_0(t)$$

the local Euler number. The reason is two-fold: first the number of vertices whose coefficient is 1 is congruent mod 2 to $\chi(K)$. Secondly when $K$ is an immersed surface in $\mathbb{R}^2$ and the ordering on the vertices is induced by projection on the $z$-axis, then the vertices with coefficient 1 are precisely the critical points of the function defined on $K$ by this projection. Thus we get that $\chi(K) = \text{number critical pts (mod 2)}$.

When $K$ is $n$-dimensional, then the ordering on the vertices induces an orientation on each simplex in $K$. Now when $s$ is an $(n-1)$-simplex and a proper face of $t$, we have that $s$ is regular in $t$ if and only if the orientation on $t$ induces the opposite orientation on $s$. Thus $\omega_{n-1}(K)$ has as a representative the sum of those $(n-1)$-simplexes whose orientation disagrees with an even number of $n$-simplexes of which it is a face (a simplex is always regular in itself). This last fact is exploited in [3].

In fact our formula can be derived from the work of Banchoff [2] and McCrory [6]. Given an ordering of the vertices of $K$ define a vertex map into $\mathbb{R}^m$ by sending the vertex $j$ into $(j, j^2, \ldots, j^m)$. This defines a full map and some combinatorics applied to the description of the S.W. classes as given in [6] gives our formula.


