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THE NUMBER OF COMPACT SUBSETS OF A TOPOLOGICAL SPACE

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ABSTRACT. Results are obtained which give an upper bound on the number of compact subsets of a topological space in terms of other cardinal invariants. The countable version of the main theorem states that an $\aleph_1$-compact space with a point-countable separating open cover has at most $2^{\aleph_0}$ compact subsets.

1. **Introduction.** Let w, L, d, c, χ, and ψ denote the following standard cardinal functions: weight, Lindelöf degree, density, cellularity, character, and pseudo-character. (For definitions, see Juhász [11].) If ϕ is a cardinal function, then $h\phi$ is the hereditary version of ϕ; i.e., $h\phi(X) = \sup\{\phi(Y): Y \subseteq X\}$. The collection of all compact subsets of a topological space X is denoted by $\mathcal{K}(X)$.

Among the best-known theorems in cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants. In this paper we will be concerned with the following such results.

I. (Pospišil [17]) If X is Hausdorff, then $|X| \leqslant 2^{2^{d(X)}}$.

II. (de Groot [6]) If X is Hausdorff, then $|X| \leqslant 2^{hL(X)}$.

III. (Arhangel′skiĭ [2]) If X is Hausdorff, then $|X| \leqslant 2^{L(X)\cdot\chi(X)}$.

IV. (Hajnal-Juhász [9]) If X is Hausdorff, then $|X| \leqslant 2^{c(X)\cdot\chi(X)}$.

V. (Hajnal-Juhász [9]) If X is T_1, then $|X| \leqslant 2^{hc(X)\cdot\psi(X)}$.

A key step in the proof of recent theorems obtained by Arhangel′skiĭ [1], Hodel [10], and Juhász [12] is to determine the number of compact subsets of a topological space. This suggests the general problem of finding an upper bound on the cardinality of $\mathcal{K}(X)$ in terms of other cardinal invariants. A particularly natural question to ask is whether or not "X" can be replaced by "$\mathcal{K}(X)$" in the inequalities in I–V. In this paper we show that this replacement is possible in I and II but not in III and IV. We do not know if the replacement is possible in V, although it is consistent in the countable case for Hausdorff spaces. In §4 we prove what is perhaps the main theorem in this paper. The countable version of this result states that the number of compact subsets of an $\aleph_1$-compact space with a point-countable separating open cover is $\leqslant 2^{\aleph_0}$.

Throughout this paper $\mathfrak{m}$ and $\mathfrak{n}$ denote cardinal numbers with $\mathfrak{m}$ infinite; $\mathfrak{m}^+$ is the smallest cardinal greater than $\mathfrak{m}$; α and β denote ordinal numbers.

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Unless otherwise stated, no separation axioms are assumed; however, regular spaces are always T_1.

We need several cardinal functions in addition to w, L, d, c, χ, and ψ. For a T_1 space X let $\psi K(X) = \aleph_0 \cdot \mathfrak{n}$, where $\mathfrak{n}$ is the smallest cardinal such that every compact subset of X is the intersection of $\leqslant \mathfrak{n}$ open sets. Let $\psi C(X)$ be the cardinal function obtained by replacing "compact" by "closed" in the definition of $\psi K(X)$. Note that $\psi K(X) \leqslant \psi C(X)$ and $\psi K(X) \leqslant hL(X)$ whenever X is Hausdorff.

The *discreteness character* of a topological space X, denoted $\Delta(X)$, is $\aleph_0 \cdot \mathfrak{n}$, where $\mathfrak{n} = \sup\{|\mathcal{F}|: \mathcal{F}$ is a discrete collection of nonempty closed sets in $X\}$. For the class of T_1 spaces this cardinal function extends the concept of $\aleph_1$-compactness (= every uncountable set has a limit point) to higher cardinality; see [10].

Let X be a set, let $\mathcal{S}$ be a cover of X. The cover $\mathcal{S}$ is said to be *separating* if given any two distinct points p and q in X, there is some S in $\mathcal{S}$ such that $p \in S$, $q \notin S$. For p in X, the *order of p with respect to* $\mathcal{S}$, denoted ord $(p, \mathcal{S})$, is the cardinality of the set $\{S$ in $\mathcal{S}$: $p \in S\}$. Now assume X is a T_1 space. The *point separating weight* of X, denoted psw (X), is $\aleph_0 \cdot \mathfrak{n}$, where $\mathfrak{n}$ is the smallest cardinal such that X has a separating open cover $\mathcal{S}$ with ord $(p, \mathcal{S}) \leqslant \mathfrak{n}$ for all p in X. Note that X has a point-countable separating open cover if and only if psw $(X) = \aleph_0$.

2. **I and II.** In this section we show that "X" can be replaced by " $\mathcal{K}(X)$ " in the inequalities in I and II.

THEOREM 2.1. *If X is a Hausdorff space, then* $|\mathcal{K}(X)| \leqslant 2^{2^{d(X)}}$.

PROOF. Let $d(X) = \mathfrak{m}$; let D be a dense subset of X with $|D| \leqslant \mathfrak{m}$. The open collection $\mathcal{S} = \{(\overline{E})^0: E \subseteq D\}$ has cardinality $\leqslant 2^{\mathfrak{m}}$; and, using the Hausdorff hypothesis and the denseness of D, one can check that $\mathcal{S}$ is a separating cover of X. Let $\mathcal{S}_1$ be all finite unions of elements of $\mathcal{S}$, and let $\mathcal{S}_2$ be all intersections of elements of $\mathcal{S}_1$. Clearly $|\mathcal{S}_2| \leqslant 2^{2^{\mathfrak{m}}}$, and so the proof is complete if we can show that $\mathcal{K}(X) \subseteq \mathcal{S}_2$. Let K be a compact subset of X. For each p in X with $p \notin K$, there is some S_p in $\mathcal{S}_1$ such that $K \subseteq S_p$ and $p \notin S_p$. Clearly $K = \bigcap\{S_p: p \in (X - K)\}$, and so $K \in \mathcal{S}_2$.

COROLLARY 2.2. *Let X be a separable Hausdorff space. Then the number of compact subsets of X is* $\leqslant 2^{2^{\aleph_0}}$.

REMARK 2.3. See [11, p. 64] for an example of a Hausdorff space of density $\mathfrak{m}$ in which the number of closed sets is $2^{2^{2^{\mathfrak{m}}}}$. If X is a regular space, then $w(X) \leqslant 2^{d(X)}$ and so the number of closed subsets of X is $\leqslant 2^{2^{d(X)}}$.

The following notation will be used in the next theorem and in §4. Let E be a set, let $\mathcal{U}$ be a collection of sets. Then $\mathcal{P}_{\mathfrak{m}}(E) = \{A \subseteq E: |A| \leqslant \mathfrak{m}\}$ and $\mathcal{U}_{\mathfrak{m}} = \{G: G$ the union of $\leqslant \mathfrak{m}$ elements of $\mathcal{U}\}$.

LEMMA 2.4 (ŠAPIROVSKIĬ [18]). *Let $\mathcal{U}$ be an open cover of a topological space X, let $\mathfrak{m} = hc(X)$. Then there is some G in $\mathcal{U}_{\mathfrak{m}}$ and some A in $\mathcal{P}_{\mathfrak{m}}(X)$ such that $X = G \cup \overline{A}$.*

THEOREM 2.5. *If X is a Hausdorff space, then* $|\mathcal{K}(X)| \leqslant 2^{hc(X) \cdot \psi K(X)}$.

PROOF. Let $\mathfrak{m} = hc(X) \cdot \psi K(X)$; then by V, $|X| \leqslant 2^{\mathfrak{m}}$. For each p in X let $\mathfrak{U}_p$ be a collection of open neighborhoods of p, closed under finite intersections, such that $|\mathfrak{U}_p| \leqslant 2^{\mathfrak{m}}$ and $\bigcap\{\overline{U}: U \text{ in } \mathfrak{U}_p\} = \{p\}$. Let $\mathfrak{U} = \bigcup\{\mathfrak{U}_p: p \text{ in } X\}$, let $\mathcal{C} = \{G \cup \overline{A}: G \text{ in } \mathfrak{U}_{\mathfrak{m}}, A \text{ in } \mathcal{P}_{\mathfrak{m}}(X)\}$, and let $\mathcal{E} = \{E: E \text{ in } \mathcal{C}_{\mathfrak{m}}\}$. Now $|\mathcal{E}| \leqslant 2^{\mathfrak{m}}$, and so the proof is complete if we can show that the complement of each compact subset of X is in $\mathcal{E}$.

Let $K \subseteq X$ be compact. Since $\psi K(X) \leqslant \mathfrak{m}$, $(X - K) = \bigcup\{F_\alpha: 0 \leqslant \alpha < \mathfrak{m}\}$, where each F_α is a closed subset of X. Fix $\alpha < \mathfrak{m}$. For each p in F_α, use the compactness of K to obtain some U_p in $\mathfrak{U}_p$ such that $U_p \cap K = \phi$. Apply Šapirovskiĭ's Lemma to F_α and $\{U_p: p \text{ in } F_\alpha\}$ to obtain G_α in $\mathfrak{U}_{\mathfrak{m}}$ and A_α in $\mathcal{P}_{\mathfrak{m}}(F_\alpha)$ such that $F_\alpha \subseteq G_\alpha \cup \overline{A}_\alpha$. Let $C_\alpha = G_\alpha \cup \overline{A}_\alpha$, and note that C_α is in $\mathcal{C}$ and $C_\alpha \cap K = \phi$. Clearly $(X - K) = \bigcup\{C_\alpha: 0 \leqslant \alpha < \mathfrak{m}\}$, and so $(X - K)$ is in $\mathcal{E}$.

COROLLARY 2.6. *Let X be a Hausdorff space which hereditarily satisfies the CCC and in which every compact set is a G_δ. Then the number of compact subsets of X is $\leqslant 2^{\aleph_0}$.*

COROLLARY 2.7. *If X is a Hausdorff space, then $|\mathcal{K}(X)| \leqslant 2^{hL(X)}$. In particular, if X is a Hausdorff, hereditarily Lindelöf space, then the number of compact subsets of X is $\leqslant 2^{\aleph_0}$.*

COROLLARY 2.8. *If X is a Hausdorff space, then $|\mathcal{K}(X)| \leqslant 2^{\Delta(X)\psi C(X)}$. In particular, if X is a Hausdorff, $\aleph_1$-compact space in which every closed set is a G_δ, then the number of compact subsets of X is $\leqslant 2^{\aleph_0}$.*

PROOF. Let $\mathfrak{m} = \Delta(X) \cdot \psi C(X)$. Then $hc(X) \leqslant \mathfrak{m}$ by a result of Hodel [10], and so $hc(X) \cdot \psi K(X) \leqslant \mathfrak{m}$. Hence $|\mathcal{K}(X)| \leqslant 2^{\mathfrak{m}}$ by 2.5.

REMARK 2.9. Hajnal and Juhász [8] have proved the consistency, for each $\mathfrak{m}$, of a regular space X with $hL(X) = \mathfrak{m}$ and $w(X) = 2^{2^{\mathfrak{m}}}$. Thus 2.5, 2.7, and 2.8 cannot be extended to "number of closed sets is $\leqslant 2^{\mathfrak{m}}$."

EXAMPLE 2.10. *A Hausdorff, hereditarily separable, non-Lindelöf space in which every closed set is a G_δ.* The example is due to the referee, and utilizes a construction of Hajnal and Juhász [7]. Let X be a set of real numbers of cardinality $\aleph_1$, and let $\mathfrak{B}$ be a countable base for the subspace Euclidean topology on X. Let X be enumerated as $\{x_\alpha: 0 \leqslant \alpha < \omega_1\}$, and for each $\alpha < \omega_1$ let $U_\alpha = \{x_\beta: \beta \leqslant \alpha\}$. Then $\mathfrak{B} \cup \{U_\alpha: 0 \leqslant \alpha < \omega_1\}$ is a subbase for a topology $\mathcal{T}$ on X which is Hausdorff, hereditarily separable, and not Lindelöf. The following lemma, due to the referee, shows that every closed subset of $(X, \mathcal{T})$ is a G_δ.

LEMMA 2.11. *Let $\mathfrak{B}$ be a countable base for a metric topology on X and let $\mathfrak{U}$ be a topology on X such that every proper open set is countable. If $\mathcal{T}$ is the topology on X generated by $\mathfrak{B} \cup \mathfrak{U}$, then every open subset of $(X, \mathcal{T})$ is an F_σ.*

PROOF. Let $H \in \mathcal{T}$; then we can write $H = G \cup D$, where G is the interior of H relative to the topology with base $\mathfrak{B}$, and $D = H - G$. Since G is an F_σ set, it suffices to show that D is countable. Suppose not. For each $x \in D$, choose $B_x \in \mathfrak{B}$ and $U_x \in \mathfrak{U}$ such that $x \in B_x \cap U_x \subseteq H$. Since $\mathfrak{B}$ is countable, there is a fixed $B \in \mathfrak{B}$ and an uncountable set $D' \subseteq D$ such that

$B_x = B$ for all $x \in D'$. Now $\bigcup_{x \in D'} U_x = X$, and so $B \subseteq H$. Hence $D' \subseteq B \subseteq G$, a contradiction.

3. **III, IV, and V.** We begin this section by giving an example of a compact, Hausdorff, first countable, separable space in which the number of compact subsets is $2^{2^{\aleph_0}}$. This example shows that "X" cannot be replaced by " $\mathcal{K}(X)$ " in the inequalities in III and IV. (The authors would like to thank I. Juhász for calling their attention to this example; see p. 68 in [11].) Let I^* be the top and bottom line of the lexicographically ordered square with the order topology, and let $X = I^* \times I^*$. It is well known that X is compact, Hausdorff, first countable, and separable. Moreover, X contains a discrete subspace of cardinality $2^{\aleph_0}$ (see [11]), and from this fact it follows that X contains $2^{2^{\aleph_0}}$ open sets. Thus, X contains $2^{2^{\aleph_0}}$ closed and hence compact sets.

We do not know if "X" can be replaced by " $\mathcal{K}(X)$ " in the inequality in V, although such a replacement is consistent in the countable case for Hausdorff spaces. The general problem seems quite difficult.

THEOREM 3.1. *Assume MA + $\neg$CH. Let X be a Hausdorff space which hereditarily satisfies the CCC and has countable pseudo-character. Then the number of compact subsets of X is $\leqslant 2^{\aleph_0}$.*

PROOF. By V, $|X| \leqslant 2^{\aleph_0}$, and so we need only show that each compact subset of X is the closure of a countable subset of X. But any compact subset of X is a compact, Hausdorff, first countable, CCC space, and by a result of Juhász [11], MA + $\neg$ CH implies that such a space is separable.

REMARK 3.2. Šapirovskiĭ [19] has recently proved that $d(X) \leqslant hc(X)^+$ whenever X is a compact Hausdorff space. Using this result, one easily obtains the following theorem. *If X is a Hausdorff space, then* $|\mathcal{K}(X)| \leqslant 2^{(\psi(X) \cdot hc(X))^+}$. Theorem 3.1 follows from this result, since MA + $\neg$ CH implies $2^{\aleph_0} = 2^{\aleph_1}$ [13].

4. **A new inequality.** Several lemmas are needed to prove the main result of this section.

LEMMA 4.1 (BURKE [3]). *Let $\mathfrak{m}$ be an infinite cardinal, let $\{E_t\colon t \text{ in } A\}$ be a collection of sets with $|A| > 2^{\mathfrak{m}}$ and $|E_t| \leqslant \mathfrak{m}$ for all t in A. Suppose $E_t = E'_t \cup E''_t$ and $E'_t \cap E''_t = \phi$ for all t in A. Then there is some $B \subseteq A$, $|B| > 2^{\mathfrak{m}}$, such that if s and t are in B, then $E'_s \cap E''_t = \phi$.*

LEMMA 4.2 (SEE [10]). *Let X be a topological space with $\Delta(X) \leqslant \mathfrak{m}$, let $\mathcal{R}$ be an open cover of X such that* ord $(p, \mathcal{R}) \leqslant \mathfrak{m}$ *for all p in X. Then there is a subcollection of $\mathcal{R}$ of cardinality $\leqslant \mathfrak{m}$ which covers X.*

LEMMA 4.3 (MIŠČENKO [5], [15]). *Let K be a set, let $\mathfrak{m}$ be an infinite cardinal, and let $\mathcal{S}$ be a collection of sets such that* ord $(p, \mathcal{S}) \leqslant \mathfrak{m}$ *for all p in K. Then the cardinality of the set of all finite minimal covers of K by elements of $\mathcal{S}$ does not exceed $\mathfrak{m}$.*

THEOREM 4.4. *If X is T_1, then* $|\mathcal{K}(X)| \leqslant 2^{\Delta(X) \cdot \mathrm{psw}(X)}$.

PROOF. Let $\mathfrak{m} = \Delta(X) \cdot \mathrm{psw}\,(X)$. First we prove that $|X| \leqslant 2^{\mathfrak{m}}$. Suppose $|X| > 2^{\mathfrak{m}}$. Let $\mathcal{S}$ be a separating open cover of X such that ord $(p, \mathcal{S}) \leqslant \mathfrak{m}$ for all p in X. For p in X let $\mathcal{S}_p = \{S \text{ in } \mathcal{S}\colon p \in S\}$. Now

$$X - \{p\} = \bigcup\{X - S\colon S \text{ in } \mathcal{S}_p\},$$

$\Delta(X - S) \leqslant \mathfrak{m}$ for each S in $\mathcal{S}_p$, and $|\mathcal{S}_p| \leqslant \mathfrak{m}$, so $\Delta(X - \{p\}) \leqslant \mathfrak{m}$. Let $\mathcal{R}_p = \{S \text{ in } \mathcal{S}\colon p \notin S\}$. Then $\mathcal{R}_p$ is an open cover of $X - \{p\}$ such that for all q in $X - \{p\}$, ord $(q, \mathcal{R}_p) \leqslant \mathfrak{m}$. By Lemma 4.2 there is a subcollection $\mathcal{W}_p$ of $\mathcal{R}_p$ with $|\mathcal{W}_p| \leqslant \mathfrak{m}$ which covers $X - \{p\}$. Note that $\mathcal{W}_p \cap \mathcal{S}_p = \phi$. By Lemma 4.1, there is some $Y \subseteq X$ with $|Y| > 2^{\mathfrak{m}}$ such that if p and q are in Y, then $\mathcal{W}_p \cap \mathcal{S}_q = \phi$. Let p and q be distinct points of Y. Pick S in $\mathcal{W}_p$ such that $q \in S$. Now $S \in \mathcal{S}_q$, and hence $\mathcal{W}_p \cap \mathcal{S}_q \neq \phi$, a contradiction. Thus we have $|X| \leqslant 2^{\mathfrak{m}}$.

Next we prove $|\mathcal{K}(X)| \leqslant 2^{\mathfrak{m}}$. Since $|X| \leqslant 2^{\mathfrak{m}}$ and ord $(p, \mathcal{S}) \leqslant \mathfrak{m}$ for all p in X, it follows that $|\mathcal{S}| \leqslant 2^{\mathfrak{m}}$. Let $\mathcal{W}$ be all finite unions of elements of $\mathcal{S}$, and let $\mathcal{G}$ be all intersections of $\leqslant \mathfrak{m}$ elements of $\mathcal{W}$. Now $|\mathcal{G}| \leqslant 2^{\mathfrak{m}}$, and so the proof is complete if we can show that $\mathcal{K}(X) \subseteq \mathcal{G}$. Given a compact subset K of X, let $\{\mathcal{S}_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}\}$ be all finite minimal covers of K by elements of $\mathcal{S}$ (use Miščenko's Lemma), and for each $\alpha < \mathfrak{m}$ let $W_\alpha = \bigcup \mathcal{S}_\alpha$. Then $K = \bigcap\{W_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}\}$ and so $K \in \mathcal{G}$.

COROLLARY 4.5. *Let X be an $\aleph_1$-compact space with a point-countable separating open cover. Then the number of compact subsets of X is $\leqslant 2^{\aleph_0}$. In particular, $|X| \leqslant 2^{\aleph_0}$.*

REMARK 4.6. In the proof of 4.4, the fact that $|\mathcal{S}| \leqslant 2^{\mathfrak{m}}$ can also be established using a technique based on ideas of M. E. Rudin [4] and Šapirovskiĭ [18]. Indeed, let $\mathcal{S}$ and X be as in 4.4, and note that it suffices to show that $d(X) \leqslant 2^{\mathfrak{m}}$. Construct transfinite sequences $\{D_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}^+\}$ and $\{\mathcal{S}_\alpha\colon 0 < \alpha < \mathfrak{m}^+\}$ such that:

(1) $|D_\alpha| \leqslant 2^{\mathfrak{m}}$ and $|\mathcal{S}_\alpha| \leqslant 2^{\mathfrak{m}}$;

(2) $\bigcup_{\beta<\alpha} D_\beta \subseteq D_\alpha \subseteq X$;

(3) $\mathcal{S}_\alpha = \{S \in \mathcal{S}\colon S \cap (\bigcup_{\beta<\alpha} D_\beta) \neq \phi\}$;

(4) if $G \in (\mathcal{S}_\alpha)_{\mathfrak{m}}$ and $G \neq X$, then $D_\alpha \cap (X - G) \neq \phi$.

Let $D = \bigcup\{D_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}^+\}$; clearly $|D| \leqslant 2^{\mathfrak{m}}$, and so it remains to prove that D is dense. Let $p \in X$, but suppose $p \notin \overline{D}$. Let $\mathcal{R} = \{S \text{ in } \mathcal{S}\colon S \cap \overline{D} \neq \phi, p \notin S\}$. Since $\mathcal{S}$ is a separating open cover of X, $\mathcal{R}$ covers $\overline{D}$. Now $\Delta(\overline{D}) \leqslant \mathfrak{m}$, so by 4.2, there is a subcollection $\mathcal{R}_1$ of $\mathcal{R}$ with $|\mathcal{R}_1| \leqslant \mathfrak{m}$ which covers $\overline{D}$. If $S \in \mathcal{R}_1$, then $S \cap D \neq \phi$. Since $D = \bigcup\{D_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}^+\}$ and $|\mathcal{R}_1| \leqslant \mathfrak{m}$, it follows that $\mathcal{R}_1 \subseteq \mathcal{S}_\alpha$ for some $\alpha < \mathfrak{m}^+$. Let $G = \bigcup \mathcal{R}_1$. Now $G \in (\mathcal{S}_\alpha)_{\mathfrak{m}}$ and $G \neq X$, so by (4), $(X - G) \cap D_\alpha \neq \phi$. This contradicts the fact that $\mathcal{R}_1$ covers $\overline{D}$.

REMARK 4.7. The above technique can also be used to prove de Groot's result that $|X| \leqslant 2^{hL(X)}$ for X Hausdorff. (This observation has also been made by R. Pol [16].) The necessary modifications are sketched as follows. Let $\mathfrak{m} = hL(X)$. For each p in X let $\mathcal{U}_p$ be a collection of open neighborhoods of p with $|\mathcal{U}_p| \leqslant \mathfrak{m}$ such that $\bigcap \mathcal{U}_p = \{p\}$. Construct transfinite sequences $\{D_\alpha\colon 0 \leqslant \alpha < \mathfrak{m}^+\}$ and $\{\mathcal{U}_\alpha\colon 0 < \alpha < \mathfrak{m}^+\}$ such that:

(1) $|D_\alpha| \leqslant 2^{\mathfrak{m}}$ and $|\mathcal{U}_\alpha| \leqslant 2^{\mathfrak{m}}$;

(2) $\bigcup_{\beta<\alpha} D_\beta \subseteq D_\alpha \subseteq X$;

(3) $\mathcal{U}_\alpha = \bigcup\{\mathcal{U}_p\colon p \in \bigcup_{\beta<\alpha} D_\beta\}$;

(4) if $G \in (\mathcal{U}_\alpha)_{\mathfrak{m}}$ and $G \neq X$, then $D_\alpha \cap (X - G) \neq \phi$.

Then $X = \bigcup\{D_\alpha : 0 \leqslant \alpha < \mathfrak{m}^+\}$ and so $|X| \leqslant 2^{\mathfrak{m}}$.

EXAMPLE 4.8. *A regular, Lindelöf space X with a σ-disjoint base which has $2^{2^{\aleph_0}}$ closed sets.* The example, due to Michael [14], is obtained as follows. Let Y be a subset of the unit interval of cardinality $2^{\aleph_0}$, all of whose compact subsets are countable. Let X be the unit interval retopologized so that Y is discrete. (Specifically, the topology is the collection of all sets of the form $U \cup V$, where U is an ordinary open set of the interval and V is a subset of Y.) This example shows that 4.5 cannot be extended to "number of closed sets is $\leqslant 2^{\aleph_0}$."

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