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ON 4-MANIFOLDS CROSS I

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ABSTRACT. In this note we set forth conditions under which for a given 4-manifold M there exist a countably infinite number of 4-manifolds M_i such that $\pi_1(\mathrm{Bd}\, M_i)$ are distinct indecomposable groups and each $M_i \times I$ is homeomorphic with $M \times I$.

1. **Introduction.** Poenaru [6] and Mazur [4] have each shown that there exists a compact 4-manifold M such that $M \times I$ is homeomorphic with $B^4 \times I$, but M is not homeomorphic with B^4. In this note we set forth conditions under which for a given 4-manifold M there exist a countably infinite number of 4-manifolds M_i such that $\pi_1(\mathrm{Bd}\, M_i)$ are distinct indecomposable groups and each $M_i \times I$ is homeomorphic with $M \times I$. An interesting aspect of this note is that we never have to compute a fundamental group.

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2. **Definitions and notations.** We let I, B^n, E^n and S^n denote the interval $[-1, 1]$, the n-ball $[-1, 1]^n$, Euclidean n-space, and the n-sphere respectively. If M is an n-manifold, then $\mathrm{Int}\, M$ and $\mathrm{Bd}\, M$ will denote the interior and boundary of M. The closure of a subset A of a topological space will be represented by $\mathrm{Cl}\, A$. We recall that a 3-manifold is *irreducible* if every embedded S^2 bounds a B^3. Let F be a surface in a 3-manifold M. If F is not a 2-sphere, then it is called *incompressible in* M if every simple closed curve on F which bounds an (open) disk in $M - F$ also bounds a disk in F.

3. **Statement of the theorem.**

THEOREM. *Suppose M is a compact 4-manifold which is obtained from the 4-mainfold N by adding a 2-handle H. If* $\mathrm{Cl}\,(\mathrm{Bd}\, M - H)$ *is an orientable, irreducible 3-manifold with incompressible boundary, then there exists a countably infinite collection of compact 4-manifolds M_i such that*:

(1) $\mathrm{Bd}\, M_i$ *is not homeomorphic with* $\mathrm{Bd}\, M_j$ *for* $i \neq j$,

(2) $\pi_1(\mathrm{Bd}\, M_i)$ *is an indecomposable group and not infinite cyclic,*

(3) $\pi_1(\mathrm{Bd}\, M_i) \neq \pi_1(\mathrm{Bd}\, M_j)$ *for* $i \neq j$ *and, hence,* $\mathrm{Int}\, M_i$ *is not homeomorphic with* $\mathrm{Int}\, M_j$,

(4) $M_i \times I$ *is homeomorphic with* $M \times I$.

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Examples of such manifolds are $(B^3 \times S^1) \cup H$ where H is a 2-handle attached to a solid torus T in $\operatorname{Bd}(B^3 \times S^1) = S^2 \times S^1$ such that T has geometric index at least 2 in $S^2 \times S^1$.

To insure condition (4) we will need the following lemmas. We define $\mathrm{id}: I \to I$ to be the identity map.

LEMMA 1. *Let $T_1, T_2 \subset \operatorname{Int} B^3$ be solid tori. Then there exists a homeomorphism $h: T_1 \to T_2$ such that $h \times \mathrm{id}: T_1 \times I \to T_2 \times I$ extends to a homeomorphism $H: B^3 \times [-2,2] \to B^3 \times [-2,2]$ which leaves the boundary of $B^3 \times [-2,2]$ pointwise fixed.*

LEMMA 2. *Let $T_1, T_2 \subset \operatorname{Bd} B^4$ be solid tori. Then there exists a homeomorphism $h: T_1 \to T_2$ such that $h \times \mathrm{id}: T_1 \times I \to T_2 \times I$ extends to a homeomorphism $H: B^4 \times I \to B^4 \times I$.*

PROOF. Lemma 1 follows from well-known unknotting techniques. From Lemma 1 we obtain a homeomorphism $h: T_1 \to T_2$ such that $h \times \mathrm{id}: T_1 \times I \to T_2 \times I$ extends to a homeomorphism $f: \operatorname{Bd}(B^4 \times I) \to \operatorname{Bd}(B^4 \times I)$ which in turn can be extended to a homeomorphism $H: B^4 \times I \to B^4 \times I$.

4. **Construction of the M_i.** We form 4-manifolds M_i by adding a pseudo 2-handle to N. Let H be the 2-handle and T_i be the regular neighborhood of the composite of i trefoil knots in $\operatorname{Bd} H$. We set T to be $N \cap H$ which is an unknotted solid torus in $\operatorname{Bd} H$. Let $h_i: T_i \to T$ be the homeomorphism promised by Lemma 2. Let $f: T \to N$ be the inclusion map. Then $M_i = N \cup_{f_i} H$ where $f_i = f \circ h_i$.

The boundary of M_i is the union of two irreducible 3-manifolds attached along a torus which is incompressible in each. Hence $\operatorname{Bd} M_i$ is irreducible.

5. **Proof of the theorem.**

(1) The 3-manifold $\operatorname{Bd} M_i$ has a Haken number [3] (maximal number of disjoint, nonparallel, incompressible surfaces) of at least $i + 1$ since there exist $i + 1$ disjoint, nonparallel, incompressible tori in $\operatorname{Bd} H - T_i$. Hence we may pick out a subsequence of the M_i (which we still denote by M_i) such that for $i \neq j$, $\operatorname{Bd} M_i$ and $\operatorname{Bd} M_j$ have distinct Haken numbers. Hence $\operatorname{Bd} M_i$ is not homeomorphic with $\operatorname{Bd} M_j$.

(2) Suppose $\pi_1(\operatorname{Bd} M_i)$ is decomposable or infinite cyclic. Then by [10] there exists an essential 2-sphere in $\operatorname{Bd} M_i$. This contradicts the irreducibility of $\operatorname{Bd} M_i$ and (2) is proved.

(3) Suppose $\pi_1(\operatorname{Bd} M_i) = \pi_1(\operatorname{Bd} M_j)$ for $i \neq j$. By the sphere theorem [5] $\pi_2(\operatorname{Bd} M_i) = 0$. Hence by well-known techniques [8] there exists a map $f: \operatorname{Bd} M_i \to \operatorname{Bd} M_j$ which induces an isomorphism on fundamental group. Since $\operatorname{Bd} M_i$ and $\operatorname{Bd} M_j$ are irreducible and contain incompressible surfaces, we can apply Waldhausen's theorem [9, Theorem 6.1, p. 77] to conclude that $\operatorname{Bd} M_i$ is homeomorphic with $\operatorname{Bd} M_j$ which is a contradiction.

Since $\pi_1(\operatorname{Bd} M_i) \neq \pi_1(\operatorname{Bd} M_j)$, we conclude [1] that $\operatorname{Int} M_i$ is not homeomorphic with $\operatorname{Int} M_j$.

(4) We define a homeomorphism h: $M_i \times I \to M \times I$ by first noting that

$$M_i \times I = N \times I \underset{f_i \times \mathrm{id}}{\cup} H \times I$$

and

$$M \times I = N \times I \underset{f \times \mathrm{id}}{\cup} H \times I.$$

We define $h|N \times I$ to be the identity. By the construction of M_i the desired homeomorphism is guaranteed.

PROPOSITION. *There exists a* 4-*manifold, namely,* Bd $(M \times I)$, *in which* M *and each* M_i *can be embedded.*

This follows since $M_i \times \{-1\} \subset \mathrm{Bd}\,(M_i \times I)$ which is homeomorphic to Bd $(M \times I)$. In fact, the double of each M_i is homeomorphic with Bd $(M \times I)$.

6. **Contractible 4-manifolds.**

DEFINITION. For each ordered n-tuple of integers $\beta = (m_1, \ldots, m_n)$ such that $\Sigma m_i = 1$, let D_β be the contractible 2-complex formed by attaching a disk D to a circle α by the formula $\alpha^{m_1}\alpha^{m_2} \cdots \alpha^{m_n}$. We call D_β a *generalized dunce hat.*

LEMMA 3. *If* M *is an* n-*dimensional manifold* $(n \geqslant 5)$ *with a generalized dunce hat as spine, then* M *is an* n-*ball.*

We first note that a generalized dunce hat can be embedded in B^3 as a spine. To see this let A_1 and A_2 be transverse annuli on a torus T such that $A_1 \cap A_2$ is a disk. Let J_1 be an arc in A_1 with endpoints in $A_1 \cap A_2$ which goes around A_1 as described by β; i.e., J_1 first winds around A_1 m_1 times, then m_2 times, etc. This is done in the most obvious manner so that J_1 can be completed to a simple closed curve J by connecting the endpoints of J_1 by an arc that runs around A_2. Now let f: $T \to S^1$ be a map which wraps T around S^1 such that the inverse of a point is either A_2 or a simple closed curve parallel to A_2. The mapping cylinder M_f of f is a solid torus with J a longitudinal curve of M_f. If care has been taken, $M_{f|J}$ is D_β minus a disk. However attaching a 2-handle to M_f along J yields B^3.

Since D_β can be embedded as a spine of B^3, by taking products, D_β can be embedded in B^n $(n \geqslant 3)$ as a spine. However by Price [7], [2, Lemma 2] there is only one way to embed D_β in E^n $(n \geqslant 5)$. Hence our lemma follows.

We are now in a position to prove Glaser's theorem [2] which states that there are uncountably many distinct contractible open 4-manifolds.

Let M be the 4-manifold M which is $B^3 \times S^1$ plus a 2-handle attached to a solid torus in Bd $(B^3 \times S^1)$ which has algebraic index 1 and geometric index not equal to 1. The manifold M has a generalized dunce hat as a spine. The proof of this fact is similar to the proof of [11, Theorem 5]. Hence $M \times I$ is homeomorphic with B^5.

Now the theorem applied to the manifold M yields a countably infinite

collection of manifolds M_i with $\pi_1(\mathrm{Bd}\ M_i)$ distinct indecomposable groups and not infinite cyclic. The existence of such a collection is the key step in Glaser's proof.

To finish the proof we form infinite sums M_α of the M_i's in uncountably many different ways such that in two different ones M_i occurs more in one than in the other for some i. Since $\pi_1(\mathrm{Bd}\, M_\alpha) \neq \pi_1(\mathrm{Bd}\, M_\beta)$ for $\alpha \neq \beta$, $\mathrm{Int}\, M_\alpha$ is not homeomorphic with $\mathrm{Int}\, M_\beta$ [1].

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