ON 4-MANIFOLDS CROSS I

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Abstract. In this note we set forth conditions under which for a given 4-manifold $M$ there exist a countably infinite number of 4-manifolds $M_i$ such that $\pi_1(Bd M_i)$ are distinct indecomposable groups and each $M_i \times I$ is homeomorphic with $M \times I$.

1. Introduction. Poenaru [6] and Mazur [4] have each shown that there exists a compact 4-manifold $M$ such that $M \times I$ is homeomorphic with $B^4 \times I$, but $M$ is not homeomorphic with $B^4$. In this note we set forth conditions under which for a given 4-manifold $M$ there exist a countably infinite number of 4-manifolds $M_i$ such that $\pi_1(Bd M_i)$ are distinct indecomposable groups and each $M_i \times I$ is homeomorphic with $M \times I$. An interesting aspect of this note is that we never have to compute a fundamental group.

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2. Definitions and notations. We let $I$, $B^n$, $E^n$ and $S^n$ denote the interval $[-1,1]$, the $n$-ball $[-1,1]^n$, Euclidean $n$-space, and the $n$-sphere respectively. If $M$ is an $n$-manifold, then Int$M$ and Bd$M$ will denote the interior and boundary of $M$. The closure of a subset $A$ of a topological space will be represented by $\text{Cl}A$. We recall that a 3-manifold is irreducible if every embedded $S^2$ bounds a $B^3$. Let $F$ be a surface in a 3-manifold $M$. If $F$ is not a 2-sphere, then it is called incompressible in $M$ if every simple closed curve on $F$ which bounds an (open) disk in $M - F$ also bounds a disk in $F$.


Theorem. Suppose $M$ is a compact 4-manifold which is obtained from the 4-manifold $N$ by adding a 2-handle $H$. If $\text{Cl}(Bd M - H)$ is an orientable, irreducible 3-manifold with incompressible boundary, then there exists a countably infinite collection of compact 4-manifolds $M_i$ such that:

1. Bd$M_i$ is not homeomorphic with Bd$M_j$ for $i \neq j$,
2. $\pi_1(Bd M_i)$ is an indecomposable group and not infinite cyclic,
3. $\pi_1(Bd M_i) \neq \pi_1(Bd M_j)$ for $i \neq j$ and, hence, Int$M_i$ is not homeomorphic with Int$M_j$,
4. $M_i \times I$ is homeomorphic with $M \times I$.

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Examples of such manifolds are \((B^3 \times S^1) \cup H\) where \(H\) is a 2-handle attached to a solid torus \(T\) in \(\text{Bd} (B^3 \times S^1) = S^2 \times S^1\) such that \(T\) has geometric index at least 2 in \(S^2 \times S^1\).

To insure condition (4) we will need the following lemmas. We define \(id: I \to I\) to be the identity map.

**Lemma 1.** Let \(T_1, T_2 \subset \text{Int} B^3\) be solid tori. Then there exists a homeomorphism \(h: T_1 \to T_2\) such that \(h \times id: T_1 \times I \to T_2 \times I\) extends to a homeomorphism \(H: B^3 \times [-2, 2] \to B^3 \times [-2, 2]\) which leaves the boundary of \(B^3 \times [-2, 2]\) pointwise fixed.

**Lemma 2.** Let \(T_1, T_2 \subset \text{Bd} B^4\) be solid tori. Then there exists a homeomorphism \(h: T_1 \to T_2\) such that \(h \times id: T_1 \times I \to T_2 \times I\) extends to a homeomorphism \(H: B^4 \times I \to B^4 \times I\).

**Proof.** Lemma 1 follows from well-known unknotting techniques. From Lemma 1 we obtain a homeomorphism \(h: T_1 \to T_2\) such that \(h \times id: T_1 \times I \to T_2 \times I\) extends to a homeomorphism \(f: \text{Bd} (B^4 \times I) \to \text{Bd} (B^4 \times I)\) which in turn can be extended to a homeomorphism \(H: B^4 \times I \to B^4 \times I\).

4. **Construction of the \(M_i\).** We form 4-manifolds \(M_i\) by adding a pseudo 2-handle to \(N\). Let \(H\) be the 2-handle and \(T_i\) be the regular neighborhood of the composite of \(i\) trefoil knots in \(\text{Bd} H\). We set \(T\) to be \(N \cap H\) which is an unknotted solid torus in \(\text{Bd} H\). Let \(h_i: T_i \to T\) be the homeomorphism promised by Lemma 2. Let \(f: T \to N\) be the inclusion map. Then \(M_i = N \cup f_i H\) where \(f_i = f \circ h_i\).

The boundary of \(M_i\) is the union of two irreducible 3-manifolds attached along a torus which is incompressible in each. Hence \(\text{Bd} M_i\) is irreducible.

5. **Proof of the theorem.**

(1) The 3-manifold \(\text{Bd} M_i\) has a Haken number \(3\) (maximal number of disjoint, nonparallel, incompressible surfaces) of at least \(i + 1\) since there exist \(i + 1\) disjoint, nonparallel, incompressible tori in \(\text{Bd} H - T\). Hence we may pick out a subsequence of the \(M_i\) (which we still denote by \(M_i\)) such that for \(i \neq j\), \(\text{Bd} M_i\) and \(\text{Bd} M_j\) have distinct Haken numbers. Hence \(\text{Bd} M_i\) is not homeomorphic with \(\text{Bd} M_j\).

(2) Suppose \(\pi_1 (\text{Bd} M_i)\) is decomposable or infinite cyclic. Then by \(10\) there exists an essential 2-sphere in \(\text{Bd} M_i\). This contradicts the irreducibility of \(\text{Bd} M_i\) and (2) is proved.

(3) Suppose \(\pi_1 (\text{Bd} M_i) = \pi_1 (\text{Bd} M_j)\) for \(i \neq j\). By the sphere theorem \(5\) \(\pi_2 (\text{Bd} M_j) = 0\). Hence by well-known techniques \(8\) there exists a map \(f: \text{Bd} M_i \to \text{Bd} M_j\) which induces an isomorphism on fundamental group. Since \(\text{Bd} M_i\) and \(\text{Bd} M_j\) are irreducible and contain incompressible surfaces, we can apply Waldhausen’s theorem \(9, \text{Theorem 6.1, p. 77}\) to conclude that \(\text{Bd} M_i\) is homeomorphic with \(\text{Bd} M_j\) which is a contradiction.

Since \(\pi_1 (\text{Bd} M_i) \neq \pi_1 (\text{Bd} M_j)\), we conclude \(1\) that \(\text{Int} M_i\) is not homeomorphic with \(\text{Int} M_j\).
(4) We define a homeomorphism \( h: M_i \times I \to M \times I \) by first noting that
\[
M_i \times I = N \times I \cup_{f_i \times \text{id}} H \times I
\]
and
\[
M \times I = N \times I \cup_{f \times \text{id}} H \times I.
\]
We define \( h|N \times I \) to be the identity. By the construction of \( M_i \) the desired homeomorphism is guaranteed.

**Proposition.** There exists a 4-manifold, namely, \( \text{Bd} (M \times I) \), in which \( M \) and each \( M_i \) can be embedded.

This follows since \( M_i \times \{-1\} \subset \text{Bd} (M \times I) \) which is homeomorphic to \( \text{Bd} (M \times I) \). In fact, the double of each \( M_i \) is homeomorphic with \( \text{Bd} (M \times I) \).

6. **Contractible 4-manifolds.**

**Definition.** For each ordered \( n \)-tuple of integers \( \beta = (m_1, \ldots, m_n) \) such that \( \Sigma m_i = 1 \), let \( D_{\beta} \) be the contractible 2-complex formed by attaching a disk \( D \) to a circle \( \alpha \) by the formula \( \alpha^{m_1} \alpha^{m_2} \cdots \alpha^{m_n} \). We call \( D_{\beta} \) a generalized dunce hat.

**Lemma 3.** If \( M \) is an \( n \)-dimensional manifold (\( n \geq 5 \)) with a generalized dunce hat as spine, then \( M \) is an \( n \)-ball.

We first note that a generalized dunce hat can be embedded in \( B^3 \) as a spine. To see this let \( A_1 \) and \( A_2 \) be transverse annuli on a torus \( T \) such that \( A_1 \cap A_2 \) is a disk. Let \( J_1 \) be an arc in \( A_1 \) with endpoints in \( A_1 \cap A_2 \) which goes around \( A_1 \) as described by \( \beta \); i.e., \( J_1 \) first winds around \( A_1 \) \( m_1 \) times, then \( m_2 \) times, etc. This is done in the most obvious manner so that \( J_1 \) can be completed to a simple closed curve \( J \) by connecting the endpoints of \( J_1 \) by an arc that runs around \( A_2 \). Now let \( f: T \to S^1 \) be a map which wraps \( T \) around \( S^1 \) such that the inverse of a point is either \( A_2 \) or a simple closed curve parallel to \( A_2 \). The mapping cylinder \( M_f \) of \( f \) is a solid torus with \( J \) a longitudinal curve of \( M_f \). If care has been taken, \( M_fJ \) is \( D_{\beta} \) minus a disk. However attaching a 2-handle to \( M_f \) along \( J \) yields \( B^3 \).

Since \( D_{\beta} \) can be embedded as a spine of \( B^3 \), by taking products, \( D_{\beta} \) can be embedded in \( B^n \) (\( n \geq 3 \)) as a spine. However by Price [7], [2, Lemma 2] there is only one way to embed \( D_{\beta} \) in \( E^n \) (\( n \geq 5 \)). Hence our lemma follows.

We are now in a position to prove Glaser's theorem [2] which states that there are uncountably many distinct contractible open 4-manifolds.

Let \( M \) be the 4-manifold \( M \) which is \( B^3 \times S^1 \) plus a 2-handle attached to a solid torus in \( \text{Bd} (B^3 \times S^1) \) which has algebraic index 1 and geometric index not equal to 1. The manifold \( M \) has a generalized dunce hat as a spine. The proof of this fact is similar to the proof of [11, Theorem 5]. Hence \( M \times I \) is homeomorphic with \( B^4 \).

Now the theorem applied to the manifold \( M \) yields a countably infinite
collection of manifolds $M_i$ with $\pi_1(\text{Bd } M_i)$ distinct indecomposable groups and not infinite cyclic. The existence of such a collection is the key step in Glaser's proof.

To finish the proof we form infinite sums $M_\alpha$ of the $M_i$'s in uncountably many different ways such that in two different ones $M_i$ occurs more in one than in the other for some $i$. Since $\pi_1(\text{Bd } M_\alpha) \neq \pi_1(\text{Bd } M_\beta)$ for $\alpha \neq \beta$, $\text{Int } M_\alpha$ is not homeomorphic with $\text{Int } M_\beta$ [1].

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