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CHARACTERIZATIONS OF $B(G)$ AND $B(G) \cap AP(G)$ FOR LOCALLY COMPACT GROUPS

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ABSTRACT. Given a locally compact (and possibly non-Abelian) group G, we denote by $B(G)$ the set of linear combinations of continuous positive-definite functions on G and by $AP(G)$ the set of continuous almost periodic functions on G. In this paper the sets $B(G)$ and $B(G) \cap AP(G)$ are characterized in terms of convolutions with measures. Specifically, let U consist of those measures $\mu \in M(G)$ for which $\|\pi(\mu)\| \leq 1$, whenever π is a continuous unitary representation of G. It is proved that a function $f \in L^\infty(G)$ belongs to (i.e. is equal locally almost everywhere to a function in) $B(G)$ if and only if the convolutions $\mu * f$, μ ranging over U, form a relatively weakly compact set in $L^\infty(G)$. The same holds if we confine our attention to either the finitely supported or the absolutely continuous measures in U. Moreover, it is shown that any of these three sets of convolutions is relatively norm compact if and only if f belongs to $B(G) \cap AP(G)$.

1. **Introduction.** Throughout this paper, G will be a locally compact (topological Hausdorff) group with a fixed left Haar measure λ (also denoted dx). The dual space of $C_0(G)$ is identified as usual with the Banach space of bounded regular complex Borel measures (with the total variation norm) on G, and denoted $M(G)$. We let $M_{dd}(G)$ stand for the linear span of the point measures δ_x, so that the norm closure of $M_{dd}(G)$ in $M(G)$ is $M_d(G)$, the set of the discrete measures in $M(G)$. With the usual norm [9, p. 141] (and the usual abuse of language), $L^\infty(G)$ denotes the Banach space of all bounded λ-measurable complex functions on G, two functions being regarded as equal if they agree locally almost everywhere (l.a.e.). Then $L^\infty(G)$ is the dual of the Banach space $L^1(G)$ of the (equivalence classes of) λ-integrable complex Borel functions on G [9, pp. 131, 148]. The Banach space $C(G)$ of bounded continuous complex functions (with the supremum norm) on G is a closed subspace of $L^\infty(G)$, and $L^1(G)$ is regarded in the usual way as a closed subspace of $M(G)$. The space of the almost periodic functions [9, p. 247] in $C(G)$ is denoted by $AP(G)$. For the definitions and basic properties of the convolutions of measures with measures or functions we refer to [9].

Suppose for a moment that G is Abelian, and Γ its dual group. Denote by $\hat{\mu}$ the Fourier-Stieltjes transform of $\mu \in M(G)$ (or $\mu \in M(\Gamma)$), and write $B(G) = \{\hat{\mu} \mid \mu \in M(\Gamma)\}$. According to a theorem of I. Kluvánek [11] a function $f \in C(G)$ is in $B(G)$ if and only if the set $\{\mu * f \mid \mu \in M_{dd}(G), \|\hat{\mu}\|_\infty \leq 1\}$ is

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relatively weakly compact in $C(G)$. C. F. Dunkl and D. E. Ramirez [7] have considered a related problem, replacing $M_{dd}(G)$ with $L^1(G)$. Their results [7, p. 453] may be rephrased as follows: A function $f \in L^\infty(G)$ is (equivalent to a function) in $B(G)$ (resp. in $\{\hat{\mu} | \mu \in M_d(\Gamma)\}$) if and only if the set $\{g * f | g \in L^1(G), \|\hat{g}\|_\infty \leqslant 1\}$ is relatively weakly compact (resp. relatively norm compact) in $L^\infty(G)$. In this paper we prove two theorems which together significantly extend these results of [7] and [11]. In fact, we consider convolutions of $f \in L^\infty(G)$ with measures in $M(G)$, $M_{dd}(G)$ or $L^1(G)$ without assuming the commutativity of G. Of course, $B(G)$, $\{\hat{\mu} | \mu \in M_d(\Gamma)\}$, and $\|\hat{\mu}\|_\infty$ must then be reinterpreted (§2). Our approach to noncommutative harmonic analysis will be the same as in P. Eymard's thesis [8], and many of our arguments rely essentially on his results.

2. **Preliminaries and notation** We refer to [3], [8], [9] for the details of the theory summarized in this section. For any function f defined on G we write $\check{f}(x) = f(x^{-1})$, ${}_af(x) = f(ax)$, $f_a(x) = f(xa)$, if $a, x \in G$. If $\mu \in M(G)$, the measures $\check{\mu}$ and $\bar{\mu}$ are defined by $\check{\mu}(f) = \mu(\check{f})$, $\bar{\mu}(f) = \overline{\mu(\bar{f})}$, $f \in C_0(G)$. Equipped with the convolution product and the involution $\mu \mapsto \mu^* = \check{\bar{\mu}}$, $M(G)$ is a Banach $*$-algebra having $L^1(G)$ as a closed $*$-ideal. If $g \in L^1(G)$ and μ is the corresponding measure, i.e. $d\mu = g\,dx$, then $d\mu^* = \Delta^{-1}\check{\bar{g}}\,dx$, where Δ is the modular function of G. We write accordingly $g^* = \Delta^{-1}\check{\bar{g}}$. The neutral element of G is denoted by e, and $C_{ru}(G)$ is the closed subspace of $C(G)$ consisting of the right uniformly continuous bounded functions [9, p. 275].

For each (strong operator) continuous unitary representation π of G on a (complex) Hilbert space H_π one obtains a $*$-representation

$$\mu \mapsto \pi(\mu) = \int \pi(s)\,d\mu(s) \in L(H_\pi)$$

of $M(G)$. Its restriction to $L^1(G)$ is essential ("non dégénérée" in [3]), and each essential $*$-representation of $L^1(G)$ is obtained in this way from a unique continuous unitary representation of G [3, p. 253]. If $\mu \in M(G)$, $\|\mu\|'$ will denote the supremum of the operator norms $\|\pi(\mu)\|$ where π ranges over all continuous unitary representations of G. Then $\|\mu\|' \leqslant \|\mu\|$ [3, p. 7]. The completion $C^*(G)$ of $L^1(G)$ with respect to the norm $g \mapsto \|g\|'$ is a C^*-algebra called the group C^*-algebra of G [3, p. 271]. Let ω: $C^*(G) \to L(H_\omega)$ denote the universal representation of $C^*(G)$ [3, p. 43]. If τ: $L^1(G) \to C^*(G)$ is the inclusion map, $\omega \circ \tau$ is an essential $*$-representation $L^1(G)$ [3, pp. 42–43]. We denote $\omega \circ \tau$ and the corresponding continuous unitary representation of G simply by ω. For all $\mu \in M(G)$, $\omega(\mu)$ belongs to the von Neumann algebra generated in $L(H_\omega)$ by $\omega(C^*(G))$ [8, p. 193]. This von Neumann algebra can be isometrically identified with the bidual $C^*(G)^{**}$ of $C^*(G)$ [3, p. 236], and so we have ω: $M(G) \to C^*(G)^{**}$. If $\mu \in M(G)$, let $\|\mu\|^\wedge$ denote the supremum of the operator norms $\|\pi(\mu)\|$ where π ranges over the continuous (topologically) *irreducible* unitary representations of G. Then $\|g\|^\wedge = \|g\|'$ $(= \|\omega(g)\|)$ if $g \in L^1(G)$ [3, pp. 40, 254–255]. From Lemma 1.23 in [8, p. 189] it thus follows that $\|\mu\|^\wedge = \|\mu\|' = \|\omega(\mu)\|$ for all $\mu \in M(G)$. In case G is Abelian, this number clearly equals $\|\hat{\mu}\|_\infty$.

Let $B(G)$ be the set of linear combinations of continuous positive-definite functions on G. By Bochner's theorem [13, p. 19] this is consistent with the customary use (as in the introduction) of the notation in the Abelian case. There is a bijection T: $B(G) \to C^*(G)^*$ satisfying $\langle Tu, g\rangle = \int g(x)u(x)\,dx$ for $u \in B(G)$, $g \in L^1(G)$ [8, p. 192]. One also gets $\langle \omega(\mu), Tu\rangle = \int u(x)\,d\mu(x)$ for $u \in B(G)$, $\mu \in M(G)$ (recall that $\omega(\mu) \in C^*(G)^{**}$) [8, p. 193]. Since $\|g\|' \leqslant \|g\|_1$ if $g \in L^1(G)$, $\|Tu\| \geqslant \|u\|_\infty$ for all $u \in B(G)$.

If E is a Banach space, the weak topology on E is by definition $\sigma(E, E^*)$ where E^* is the (topological) dual space of E. Any norm closed linear subspace F of E is also weakly closed, and $\sigma(E, E^*)$ restricted to F agrees with $\sigma(F, F^*)$.

3. **Characterizations of** $B(G)$. The following lemma contains generalizations and analogues of some results in [6, p. 503]. The basic technique in our proof is the same as in [6]. We are indebted to the referee for pointing out that the equivalence of the relative norm compactness of the sets A and B has been proved, in the Abelian case, by J. W. Kitchen [10, p. 235].

LEMMA 3.1. *Let G be a locally compact group and $f \in L^\infty(G)$. The set $A = \{\delta_x * f | x \in G\}$ is relatively compact in the weak (resp. norm) topology of $L^\infty(G)$ if and only if $B = \{g * f | g \in L^1(G),\ \|g\|_1 \leqslant 1\}$ is so. If any one of these conditions is satisfied, then f is equal l.a.e. to a function in $C_{ru}(G)$.*

PROOF. First we show that if B is relatively weakly compact, then f is equal l.a.e. to a function in $C_{ru}(G)$. Let $(g_i)_{i\in\mathfrak{I}}$ be an approximate unit (of norm one) in $L^1(G)$ [9, p. 303]. For all $g \in L^1(G)$ we have

$$\langle f, g\rangle = \int g(x)f(x)\,dx = \int \Delta(x^{-1})g(x^{-1})f(x^{-1}e)\,dx = \overline{g^*} * f(e)$$
$$= \lim_i (\overline{g^*} * g_i) * f(e) = \lim_i \overline{g^*} * (g_i * f)(e) = \lim_i \langle g_i * f, g\rangle.$$

Since $L^1(G) * l^\infty(G) \subset C_{ru}(G)$ [9, p. 295], the values of the above convolutions at e are well defined, and $g_i * f \in C_{ru}(G)$ for all $i \in \mathfrak{I}$. As $C_{ru}(G)$ is weakly closed in $L^\infty(G)$, the net $(g_i * f)_{i\in\mathfrak{I}}$ has by assumption a subnet which converges weakly, hence in $\sigma(L^\infty, L^1)$, to a function in $C_{ru}(G)$. By the above calculation this function must be l.a.e. equal to f. Next, let X be the Stone-Čech compactification of G, i.e. the maximal ideal space of the commutative C^*-algebra $C(G)$. We identify G as usual with a dense subset of X. Define S: $L^1(G) \to C(G) = C(X)$ by $Sg = g * f$, and let σ: $X \to L^\infty(G)$ be the map satisfying

$$\langle \sigma(x), g\rangle = Sg(x).$$

It is known that σ is always $\sigma(L^\infty, L^1)$-continuous, and that it is weakly (resp. norm) continuous if and only if S is a weakly compact operator (resp. a compact operator) [4, p. 490]. Suppose now that $\sigma(G)$ is contained in a weakly (resp. norm) compact, hence $\sigma(L^\infty, 1^1)$-compact set D. Then $\sigma(X) \subset D$, because G is dense in X. As $\sigma(L^\infty, L^1)$ and the weak (resp. norm) topology agree on D, σ is weakly (resp. norm) continuous, and so S is a weakly compact operator (resp. a compact operator). Conversely, if S is a weakly compact operator (resp. a compact operator), then $\sigma(G)$ is contained in the weakly (resp. norm) compact set $\sigma(X)$. Now, $\sigma(x)$ is simply ${}_{x^{-1}}\check{f}\,(=\delta_x * \check{f})$ for all $x \in G$. In fact, for $g \in L^1(G)$ we have

$$\langle \sigma(x), g\rangle = \int g(y) f(y^{-1}x)\, dy = \int {}_{x^{-1}}\check{f}\,(y) g(y)\, dy = \langle {}_{x^{-1}}\check{f}, g\rangle.$$

Suppose now that B is relatively weakly compact, i.e. S is a weakly compact operator. By the first part of the proof we may assume $f \in C(G)$. Since $\{{}_{x^{-1}}\check{f} | x \in G\}$ is relatively weakly compact in $L^\infty(G)$, and $\check{f}$ is continuous, the set $A = \{{}_{x^{-1}}f | x \in G\}$, i.e. the image of $\{\check{f}_x | x \in G\}$ under the isometric [9, p. 295] linear isomorphism $g \mapsto \check{g}$, is also relatively weakly compact by virtue of a result due to A. Grothendieck (see e.g. [12, p. 91]). Suppose, conversely, that A is relatively weakly compact. Then so is $\{g * \check{f} | g \in L^1(G),\ \|g\|_1 \leqslant 1\}$. Therefore $\check{f}$ and f may be assumed to be continuous. By replacing $\check{f}$ with f in the preceding argument we see that $\{{}_{x^{-1}}\check{f} | x \in G\}$ is relatively weakly compact, so that S is a weakly compact operator. The proof that the relative norm compactness of A is equivalent to that of B is similar, but in passing from $\{{}_{x^{-1}}\check{f} | x \in G\}$ to A this time use is made of the equivalence of left and right almost periodicity (see e.g. Theorem 18.1 in [9, p. 246]).

THEOREM 3.2. *For a locally compact group G and $f \in L^\infty(G)$ the following four conditions are equivalent*:

(i) *f is equal l.a.e. to a function in $B(G)$*,

(ii) $\{\mu * f | \mu \in M(G),\ \|\mu\|^\wedge \leqslant 1\}$ *is relatively weakly compact in* $L^\infty(G)$,

(iii) $\{\mu * f | \mu \in M_{dd}(G),\ \|\mu\|^\wedge \leqslant 1\}$ *is relatively weakly compact in* $L^\infty(G)$,

(iv) $\{g * f | g \in L^1(G),\ \|g\|^\wedge \leqslant 1\}$ *is relatively weakly compact in* $L^\infty(G)$.

PROOF. Let us first show that (i) implies (ii). We denote $A = C^*(G)^{**}$, $A_* = C^*(G)^*$, identify A_* with its canonical image in A^*, and write ${}_a g(x) = g(ax)$ for $a, x \in A$, $g \in A^*$. Since A is a C^*-algebra, the two Arens products in A^{**} coincide [2, p. 869]. From Theorem 4.2 in [12, p. 91] we see therefore (taking $E = \{x \in A |\ \|x\| \leqslant 1\}$ and $\mathfrak{F} = A^*|E$ in the notation of [12]) that for all $g \in A^*$, $\{{}_a g | a \in A,\ \|a\| \leqslant 1\}$ is a relatively weakly compact subset of A^*. Alternatively, one may apply Corollary II.9 in [1, p. 293] which shows that an arbitrary bounded linear map from a C^*-algebra to A^* is a weakly compact operator. If $g \in A_*$, $\{{}_a g | a \in A,\ \|a\| \leqslant 1\}$ is a relatively weakly (i.e. $\sigma(A_*, A)$) compact subset of A_*, because the multiplication in A is separately continuous in the weak operator topology, which coincides with $\sigma(A, A_*)$ on A $(\subset L(H_\omega))$ [3, p. 237]. Let $T\colon B(G) \to A_*$ be as in §2. Assume now $f \in B(G)$. We show that ${}_{\omega(\mu)}(Tf) = T(\check{\mu} * f)$ (note that $\check{\mu} * f \in B(G)$ [8, p. 198]; this also follows from the argument below). Since ${}_{\omega(\mu)}(Tf) \in A_*$, ${}_{\omega(\mu)}(Tf) = Th$ for some $h \in B(G)$. Thus we have (l.a.e.)

$$\begin{aligned} h(x) &= \langle {}_{\omega(\mu)}(Tf), \omega(\delta_x)\rangle = \langle Tf, \omega(\mu * \delta_x)\rangle = \int f\, d(\mu * \delta_x) \\ &= \int f(sx)\, d\mu(s) = \int f(s^{-1}x)\, d\check{\mu}(s) = \check{\mu} * f(x). \end{aligned}$$

In [8, p. 197] it is observed that $B(G)$ is invariant under complex conjugation, and $\|Tu\| = \|T\bar{u}\|$ for all $u \in B(G)$. Therefore

$$\|\check{\mu}\|^\wedge = \|(\check{\mu})^*\|^\wedge = \|\bar{\mu}\|^\wedge = \sup\left\{\left|\int \bar{u}(x)\, d\bar{\mu}(x)\right| \,|u \in B(G),\ \|Tu\| \leqslant 1\right\}$$

$$= \sup\left\{\left|\int u(x)\, d\mu(x)\right| \,|u \in B(G),\ \|Tu\| \leqslant 1\right\} = \|\mu\|^\wedge, \qquad \mu \in M(G).$$

After these preparations it is clear that since the linear map $S\colon A_* \to L^\infty(G)$ for which $S(Tu) = u$, $u \in B(G)$, is norm decreasing, hence weakly continuous, the set

$$\{\mu * f | \mu \in M(G),\ \|\mu\|^\wedge \leqslant 1\} = \{\check{\mu} * f | \mu \in M(G),\ \|\mu\|^\wedge \leqslant 1\}$$
$$= \{S({}_{\omega(\mu)}(Tf)) | \mu \in M(G),\ \|\omega(\mu)\| \leqslant 1\}$$

is relatively weakly compact in $L^\infty(G)$. Obviously, (ii) implies both (iii) and (iv). Suppose now that (iii) holds. In view of the preceding lemma f may be assumed to be continuous. Let G_d denote G equipped with the discrete topology. Then $M_d(G)$ can be identified with $M(G_d) = L^1(G_d)$. Composed with the identity map of G_d onto G, each continuous unitary representation π of G defines a unitary representation π_d of G_d, and $\pi(\mu) = \pi_d(\mu)$ for all $\mu \in M(G_d)$. Thus, if $\|\mu\|'_d$ for $\mu \in M(G_d)$ denotes the supremum of $\|\pi(\mu)\|$ over the unitary representations π of G_d, then $\|\mu\|'_d \geqslant \|\mu\|^\wedge$. Since $\|\mu_n - \mu\|^\wedge \to 0$ if $\mu_n \to \mu$ in total variation norm, we get

$$\sup\left\{\left|\int \check{f}\, d\mu\right| \,|\, \mu \in L^1(G_d),\ \|\mu\|_d' \leqslant 1\right\}$$
$$\leqslant \sup\{|\mu * f(e)| \,|\, \mu \in M_{dd}(G),\ \|\mu\|^\wedge \leqslant 1\}.$$

The latter quantity is finite by assumption, because evaluation at e is a continuous functional on $C(G)$. Therefore $\check{f} \in B(G_d)$ [8, p. 191], and so also $f \in B(G_d)$. As f is continuous on G, $f \in B(G)$ [8, p. 202]. Finally, assume (iv). Since $\|g\|' \leqslant \|g\|_1$ for $g \in L^1(G)$, Lemma 3.1 shows that f may be taken to be continuous. As $g * f \in C(G)$ and $\int \check{f}(x) g(x)\, dx = g * f(e)$ for $g \in L^1(G)$, we can again apply Proposition 2.1 in [8, p. 191] and conclude that $\check{f} \in B(G)$. Thus $f \in B(G)$, i.e. (i) holds.

4. **Characterizations of** $B(G) \cap AP(G)$.

THEOREM 4.1. *For a locally compact group G and $f \in L^\infty(G)$ the following four conditions are equivalent*:

(i) *f is equal l.a.e. to a function in $B(G) \cap AP(G)$*,

(ii) $\{\mu * f | \mu \in M(G),\ \|\mu\|^\wedge \leqslant 1\}$ *is relatively compact in $L^\infty(G)$*,

(iii) $\{\mu * f | \mu \in M_{dd}(G),\ \|\mu\|^\wedge \leqslant 1\}$ *is relatively compact in $L^\infty(G)$*,

(iv) $\{g * f | g \in L^1(G),\ \|g\|^\wedge \leqslant 1\}$ *is relatively compact in $L^\infty(G)$*.

PROOF. To prove that (i) implies (ii) we may assume that $f \in B(G) \cap AP(G)$. Let us first treat the special case where G is compact. We take [5] as our general reference on the representation theory of compact groups. Let $\hat{G}$ denote the dual of G, i.e. the set of all equivalence classes of continuous irreducible unitary representations of G, and choose from each class $\alpha \in \hat{G}$ a fixed representation T_α of G on an n_α-dimensional Hilbert space H_α (as G is compact, $n_\alpha < \infty$). Denote by $L^\infty(\hat{G})$ the space of families $\varphi = (\varphi_\alpha)_{\alpha \in \hat{G}}$ where $\varphi_\alpha \in L(H_\alpha)$ and $\|\varphi\|_\infty = \sup\{\|\varphi_\alpha\| \mid \alpha \in \hat{G}\} < \infty$ ($\|\varphi_\alpha\|$ is the usual operator norm). Let $L^1(\hat{G})$ consist of all $\varphi \in L^\infty(G)$ satisfying $\|\varphi\|_1 = \sum_\alpha n_\alpha \operatorname{Tr}(\varphi_\alpha^* \varphi_\alpha)^{1/2} < \infty$, and let $C_0(\hat{G})$ be the set of those $\varphi \in L^\infty(\hat{G})$ for which the function $\alpha \mapsto \|\varphi_\alpha\|$ vanishes at infinity on the discrete space $\hat{G}$. Then $C_0(\hat{G})$ and $L^\infty(\hat{G})$ are C^*-algebras with coordinatewise operations, and $L^1(\hat{G}) = C_0(\hat{G})^*$, $L^\infty(\hat{G}) = L^1(\hat{G})^*$, the duality being implemented by $\langle \varphi, \psi \rangle = \sum_\alpha n_\alpha \operatorname{Tr}(\varphi_\alpha \psi_\alpha)$ for $\psi \in L^1(\hat{G})$, $\varphi \in C_0(\hat{G})$ or $\varphi \in L^\infty(\hat{G})$ [5, p. 88]. If $\psi \in L^1(\hat{G})$ and $\varphi \in L^\infty(\hat{G})$, then $\psi\varphi \in L^1(\hat{G})$ and $\|\psi\varphi\|_1 \leqslant \|\psi\|_1 \|\varphi\|_\infty$. Thus the map $M_\psi\colon L^\infty(\hat{G}) \to L^1(\hat{G})$, $M_\psi(\varphi) = \psi\varphi$, is for each $\psi \in L^1(\hat{G})$ a bounded linear operator, and $\|M_\psi\| \leqslant \|\psi\|_1$. If $\{\alpha \in \hat{G} \mid \psi_\alpha \neq 0\}$ is finite, the range of M_ψ is finite dimensional. Since such elements ψ are dense in $L^1(\hat{G})$, it follows that M_ψ is a compact operator for all $\psi \in L^1(\hat{G})$. It is well known that $C^*(G)$ is isometrically $*$-isomorphic to $C_0(\hat{G})$. (In fact, the Fourier transformation [3, p. 316] $\mathfrak{F}\colon L^1(G) \to C_0(\hat{G})$ is an injective $*$-algebra homomorphism, $\mathfrak{F}(L^1(G))$ is dense in $C_0(\hat{G})$ [5, p. 90], and $\|\mathfrak{F}(g)\|_\infty = \|g\|'$ [3, pp. 40, 254–255].) As the canonical embedding of $C_0(\hat{G})$ into its bidual can be interpreted as the inclusion of $C_0(\hat{G})$ into $L^\infty(\hat{G})$, and $M_\psi\varphi = {}_\varphi\psi$ for $\psi \in L^1(\hat{G})$, $\varphi \in L^\infty(\hat{G})$ (where as usual $\langle {}_\varphi\psi, \varphi' \rangle = \langle \psi, \varphi\varphi' \rangle$, $\varphi' \in L^\infty(\hat{G})$), the set $\{{}_a g \mid a \in C^*(G)^{**}, \|a\| \leqslant 1\}$ is therefore relatively compact in $C^*(G)^*$ for all $g \in C^*(G)^*$. In the case of a compact group G, the proof that (i) implies (ii) is now easily completed by the same reasoning as the corresponding part in the proof of Theorem 3.2. In the general case, let $\overline{G}$ denote the almost periodic compactification of G [3, p. 297], and $\sigma\colon G \to \overline{G}$ the canonical homomorphism. As $f \in B(G) \cap AP(G)$, there is a unique $\tilde{f} \in B(\overline{G})$ satisfying $f = \tilde{f} \circ \sigma$ [8, p. 203]. For $\mu \in M(G)$, let $\tilde{\mu} \in M(\overline{G})$ be defined by $\langle \tilde{\mu}, g \rangle = \int g \circ \sigma(x)\, d\mu(x)$, $g \in C(\overline{G})$. It is easily verified that $(\tilde{\mu} * \tilde{f}) \circ \sigma = \mu * f$ ($\in B(G) \subset C(G)$, see the proof of Theorem 3.2). Furthermore, if $\pi\colon \overline{G} \to L(H_\pi)$ is any continuous unitary representation, and $\xi, \eta \in H_\pi$, then

$$\int (\pi \circ \sigma(x)\xi, \eta)\, d\mu(x) = \int (\pi(y)\xi, \eta)\, d\tilde{\mu}(y),$$

so that $\pi \circ \sigma(\mu) = \pi(\tilde{\mu})$. This shows that $\|\tilde{\mu}\|' \leqslant \|\mu\|'$, and so

$$\{\mu * f \mid \mu \in M(G), \|\mu\|' \leqslant 1\} \subset \{(\tilde{\mu} * \tilde{f}) \circ \sigma \mid \mu \in M(G), \|\tilde{\mu}\|' \leqslant 1\}$$
$$\subset \left\{(\nu * \tilde{f}) \circ \sigma \mid \nu \in M(\overline{G}), \|\nu\|' \leqslant 1\right\}.$$

Recalling the first part of the proof we now see that, since $g \mapsto g \circ \sigma$ is an isometry from $C(\overline{G})$ into $C(G)$, even in the general case (i) implies (ii). Clearly, (ii) implies both (iii) and (iv). Finally, if we assume either (iii) or (iv), then f is by Lemma 3.1 equal l.a.e. to a function in $AP(G)$. By Theorem 3.2 this function belongs to $B(G)$, and so (i) holds.

REMARK. If G is a locally compact Abelian group and Γ its dual group, then $\|\mu\|^\wedge = \|\hat{\mu}\|_\infty$ for $\mu \in M(G)$, $B(G) = \{\hat{\nu} \mid \nu \in M(\Gamma)\}$ (§2), and $B(G) \cap AP(G) = \{\hat{\nu} \mid \nu \in M_{dd}(\Gamma)\}$ (see e.g. [8, p. 204]). Therefore the theorem in [11, p. 84] and Theorem 4 in [7, p. 453] follow from Theorem 3.2, and Theorem 5 in [7, p. 453] follows from Theorem 4.1.

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