CHARACTERIZATIONS OF $B(G)$ AND $B(G) \cap AP(G)$ FOR LOCALLY COMPACT GROUPS

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Abstract. Given a locally compact (and possibly non-Abelian) group $G$, we denote by $B(G)$ the set of linear combinations of continuous positive-definite functions on $G$ and by $AP(G)$ the set of continuous almost periodic functions on $G$. In this paper the sets $B(G)$ and $B(G) \cap AP(G)$ are characterized in terms of convolutions with measures. Specifically, let $U$ consist of those measures $\mu \in M(G)$ for which $\|\pi(\mu)\| < 1$, whenever $\pi$ is a continuous unitary representation of $G$. It is proved that a function $f \in L^\infty(G)$ belongs to (i.e. is equal locally almost everywhere to a function in) $B(G)$ if and only if the convolutions $\mu * f$, $\mu$ ranging over $U$, form a relatively weakly compact set in $L^\infty(G)$. The same holds if we confine our attention to either the finitely supported or the absolutely continuous measures in $U$. Moreover, it is shown that any of these three sets of convolutions is relatively norm compact if and only if $f$ belongs to $B(G) \cap AP(G)$.

1. Introduction. Throughout this paper, $G$ will be a locally compact (topological Hausdorff) group with a fixed left Haar measure $\lambda$ (also denoted $dx$). The dual space of $C_0(G)$ is identified as usual with the Banach space of bounded regular complex Borel measures (with the total variation norm) on $G$, and denoted $M(G)$. We let $M_{dd}(G)$ stand for the linear span of the point measures $\delta_x$, so that the norm closure of $M_{dd}(G)$ in $M(G)$ is $M_d(G)$, the set of the discrete measures in $M(G)$. With the usual norm [9, p. 141] (and the usual abuse of language), $L^\infty(G)$ denotes the Banach space of all bounded $\lambda$-measurable complex functions on $G$, two functions being regarded as equal if they agree locally almost everywhere (l.a.e.). Then $L^\infty(G)$ is the dual of the Banach space $L^1(G)$ of the (equivalence classes of) $\lambda$-integrable complex Borel functions on $G$ [9, pp. 131, 148]. The Banach space $C(G)$ of bounded continuous functions (with the supremum norm) on $G$ is a closed subspace of $L^\infty(G)$, and $L^1(G)$ is regarded in the usual way as a closed subspace of $M(G)$. The space of the almost periodic functions [9, p. 247] in $C(G)$ is denoted by $AP(G)$. For the definitions and basic properties of the convolutions of measures with measures or functions we refer to [9].

Suppose for a moment that $G$ is Abelian, and $\Gamma$ its dual group. Denote by $\hat{\mu}$ the Fourier-Stieltjes transform of $\mu \in M(G)$ (or $\mu \in M(\Gamma)$), and write $B(G) = \{ \hat{\mu} | \mu \in M(\Gamma) \}$. According to a theorem of I. Kluvánek [11] a function $f \in C(G)$ is in $B(G)$ if and only if the set $\{ \mu \ast f | \mu \in M_{dd}(G), \| \hat{\mu} \|_\infty \leq 1 \}$ is...
relatively weakly compact in $C(G)$. C. F. Dunkl and D. E. Ramirez [7] have considered a related problem, replacing $M_{dd}(G)$ with $L'(G)$. Their results [7, p. 453] may be rephrased as follows: A function $f \in L^\infty(G)$ is (equivalent to a function) in $B(G)$ (resp. in $\{ \hat{\mu} \mid \mu \in M_d(\Gamma) \}$) if and only if the set \{ $g \ast f \mid g \in L^1(G), \| \hat{g} \|_\infty \leq 1$ \} is relatively weakly compact (resp. relatively norm compact) in $L^\infty(G)$. In this paper we prove two theorems which together significantly extend these results of [7] and [11]. In fact, we consider convolutions of $f \in L^\infty(G)$ with measures in $M(G)$, $M_{dd}(G)$ or $L'(G)$ without assuming the commutativity of $G$. Of course, $B(G)$, \{ $\hat{\mu} \mid \mu \in M_d(\Gamma) \}$, and $\| \hat{\mu} \|_\infty$ must then be reinterpreted (§2). Our approach to noncommutative harmonic analysis will be the same as in P. Eymard’s thesis [8], and many of our arguments rely essentially on his results.

2. Preliminaries and notation

We refer to [3], [8], [9] for the details of the theory summarized in this section. For any function $f$ defined on $G$ we write $\tilde{f}(x) = f(x^{-1})$, $af(x) = f(ax)$, $a_f(x) = f(xa)$, if $a, x \in G$. If $\mu \in M(G)$, the measures $\hat{\mu}$ and $\hat{\mu}'$ are defined by $\hat{\mu}(f) = \mu(\tilde{f})$, $\hat{\mu}'(f) = \mu'(f)$, $f \in C_0(G)$. Equipped with the convolution product and the involution $\mu \mapsto \mu^* = \hat{\mu}$, $M(G)$ is a Banach *-algebra having $L^1(G)$ as a closed *-ideal. If $g \in L^1(G)$ and $\mu$ is the corresponding measure, i.e. $d\mu = g \, dx$, then $d\mu^* = \Delta^{-1/2} g \, dx$, where $\Delta$ is the modular function of $G$. We write accordingly $g^* = \Delta^{-1/2} \hat{g}$. The neutral element of $G$ is denoted by $e$, and $C_m(G)$ is the closed subspace of $C(G)$ consisting of the right uniformly continuous bounded functions [9, p. 275].

For each (strong operator) continuous unitary representation $\pi$ of $G$ on a (complex) Hilbert space $H_\pi$ one obtains a *-representation

$$\mu \mapsto \pi(\mu) = \int \pi(s) \, d\mu(s) \in L(H_\pi)$$

of $M(G)$. Its restriction to $L^1(G)$ is essential (“non dégénérée” in [3]), and each essential *-representation of $L^1(G)$ is obtained in this way from a unique continuous unitary representation of $G$ [3, p. 253]. If $\mu \in M(G)$, $\| \mu \|'$ will denote the supremum of the operator norms $\| \pi(\mu) \|$ where $\pi$ ranges over all continuous unitary representations of $G$. Then $\| \mu \|' \leq \| \mu \|$ [3, p. 7]. The completion $C^*(G)$ of $L^1(G)$ with respect to the norm $g \mapsto \| g \|'$ is a C*-algebra called the group C*-algebra of $G$ [3, p. 271]. Let $\omega: C^*(G) \to L(H_\omega)$ denote the universal representation of $C^*(G)$ [3, p. 43]. If $\tau: L^1(G) \to C^*(G)$ is the inclusion map, $\omega \circ \tau$ is an essential *-representation $L^1(G)$ [3, pp. 42–43]. We denote $\omega \circ \tau$ and the corresponding continuous unitary representation of $G$ simply by $\omega$. For all $\mu \in M(G)$, $\omega(\mu)$ belongs to the von Neumann algebra generated in $L(H_\omega)$ by $\omega(C^*(G))$ [8, p. 193]. This von Neumann algebra can be isometrically identified with the bidual $C^*(G)^{**}$ of $C^*(G)$ [3, p. 236], and so we have $\omega: M(G) \to C^*(G)^{**}$. If $\mu \in M(G)$, let $\| \mu \|''$ denote the supremum of the operator norms $\| \pi(\mu) \|$ where $\pi$ ranges over the continuous (topologically) irreducible unitary representations of $G$. Then $\| g \|'' = \| g \|'$ ($= \| \omega(g) \|$) if $g \in L^1(G)$ [3, pp. 40, 254–255]. From Lemma 1.23 in [8, p. 189] it thus follows that $\| \mu \|'' = \| \mu \|' = \| \omega(\mu) \|$ for all $\mu \in M(G)$. In case $G$ is Abelian, this number clearly equals $\| \hat{\mu} \|_\infty$. License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Let $B(G)$ be the set of linear combinations of continuous positive-definite functions on $G$. By Bochner’s theorem [13, p. 19] this is consistent with the customary use (as in the introduction) of the notation in the Abelian case. There is a bijection $T: B(G) \rightarrow C^\ast(G)^\ast$ satisfying $\langle Tu, g \rangle = \int g(x)u(x) \, dx$ for $u \in B(G), g \in L^1(G)$ [8, p. 192]. One also gets $\langle \omega(\mu), Tu \rangle = \int u(x) \, d\mu(x)$ for $u \in B(G), \mu \in M(G)$ (recall that $\omega(\mu) \in C^\ast(G)^{\ast\ast}$) [8, p. 193]. Since $\|u\|^\ast < \|u\|_1$ if $g \in L^1(G), \|Tu\| > \|u\|_\infty$ for all $u \in B(G)$.

If $E$ is a Banach space, the weak topology on $E$ is by definition $\sigma(E, E^\ast)$ where $E^\ast$ is the (topological) dual space of $E$. Any norm closed linear subspace $F$ of $E$ is also weakly closed, and $\sigma(E, E^\ast)$ restricted to $F$ agrees with $\sigma(F, F^\ast)$.

3. Characterizations of $B(G)$. The following lemma contains generalizations and analogues of some results in [6, p. 503]. The basic technique in our proof is the same as in [6]. We are indebted to the referee for pointing out that the equivalence of the relative norm compactness of the sets $A$ and $B$ has been proved, in the Abelian case, by J. W. Kitchen [10, p. 235].

Lemma 3.1. Let $G$ be a locally compact group and $f \in L^\infty(G)$. The set $A = \{g \ast f \mid g \in G\}$ is relatively compact in the weak (resp. norm) topology of $L^\infty(G)$ if and only if $B = \{g \ast f \mid g \in L^1(G), \|g\|_1 < 1\}$ is so. If any one of these conditions is satisfied, then $f$ is equal l.a.e. to a function in $C_m(G)$.

Proof. First we show that if $B$ is relatively weakly compact, then $f$ is equal l.a.e. to a function in $C_m(G)$. Let $(g_i)_{i \in \Gamma}$ be an approximate unit (of norm one) in $L^1(G)$ [9, p. 303]. For all $g \in L^1(G)$ we have

$$\langle f, g \rangle = \int g(x)f(x) \, dx = \int \Delta(x^{-1}) g(x^{-1})f(x^{-1}e) \, dx = \overline{g}^\ast \ast f(e) = \lim_i (g_i^\ast \ast g_i) \ast f(e) = \lim_i g_i^\ast \ast (g_i \ast f)(e) = \lim \langle g_i \ast f, g \rangle.$$ 

Since $L^1(G) \ast L^\infty(G) \subset C_m(G)$ [9, p. 295], the values of the above convolutions at $e$ are well defined, and $g_i \ast f \in C_m(G)$ for all $i \in \Gamma$. As $C_m(G)$ is weakly closed in $L^\infty(G)$, the net $(g_i \ast f)_{i \in \Gamma}$ has by assumption a subnet which converges weakly, hence in $\sigma(L^\infty, L^1)$, to a function in $C_m(G)$. By the above calculation this function must be l.a.e. equal to $f$. Next, let $X$ be the Stone-Cech compactification of $G$, i.e. the maximal ideal space of the commutative $C^\ast$-algebra $C(G)$. We identify $G$ as usual with a dense subset of $X$. Define $S: L^1(G) \rightarrow C(G) = C(X)$ by $Sg = g \ast f$, and let $\sigma: X \rightarrow L^\infty(G)$ be the map satisfying

$$\langle \sigma(x), g \rangle = Sg(x).$$

It is known that $\sigma$ is always $\sigma(L^\infty, L^1)$-continuous, and that it is weakly (resp. norm) continuous if and only if $S$ is a weakly compact operator (resp. a compact operator) [4, p. 490]. Suppose now that $\sigma(G)$ is contained in a weakly (resp. norm) compact, hence $\sigma(L^\infty, L^1)$-compact set $D$. Then $\sigma(X) \subset D$, because $G$ is dense in $X$. As $\sigma(L^\infty, L^1)$ and the weak (resp. norm) topology agree on $D$, $\sigma$ is weakly (resp. norm) continuous, and so $S$ is a weakly compact operator (resp. a compact operator). Conversely, if $S$ is a weakly compact operator (resp. a compact operator), then $\sigma(G)$ is contained in the
weakly (resp. norm) compact set $\sigma(X)$. Now, $\sigma(x)$ is simply $x^{-1}f (= \delta_x \ast f)$ for all $x \in G$. In fact, for $g \in L^1(G)$ we have
\[ \langle \sigma(x), g \rangle = \int g(y) f(y^{-1}x) \, dy = \int x^{-1}f(y) g(y) \, dy = \langle x^{-1}f, g \rangle. \]

Suppose now that $B$ is relatively weakly compact, i.e. $S$ is a weakly compact operator. By the first part of the proof we may assume $f \in C(G)$. Since $\{x^{-1}f \mid x \in G\}$ is relatively weakly compact in $L^\infty(G)$, and $f$ is continuous, the set $A = \{x^{-1}f \mid x \in G\}$, i.e. the image of $\{\hat{f} \mid x \in G\}$ under the isometric [9, p. 295] linear isomorphism $g \mapsto \hat{g}$, is also relatively weakly compact by virtue of a result due to A. Grothendieck (see e.g. [12, p. 91]). Suppose, conversely, that $A$ is relatively weakly compact. Then so is $\{g \ast \hat{f} \mid g \in L^1(G), \|g\|_1 < 1\}$. Therefore $\hat{f}$ and $f$ may be assumed to be continuous. By replacing $\hat{f}$ with $f$ in the preceding argument we see that $\{x^{-1}f \mid x \in G\}$ is relatively weakly compact, so that $S$ is a weakly compact operator. The proof that the relative norm compactness of $A$ is equivalent to that of $B$ is similar, but in passing from $\{x^{-1}f \mid x \in G\}$ to $A$ this time use is made of the equivalence of left and right almost periodicity (see e.g. Theorem 18.1 in [9, p. 246]).

**Theorem 3.2.** For a locally compact group $G$ and $f \in L^\infty(G)$ the following four conditions are equivalent:

(i) $f$ is equal l.a.e. to a function in $B(G)$,

(ii) $\{\mu \ast f \mid \mu \in M_d(G), \|\mu\| < 1\}$ is relatively weakly compact in $L^\infty(G)$,

(iii) $\{\mu \ast f \mid \mu \in M((G), \|\mu\| < 1\}$ is relatively weakly compact in $L^\infty(G)$,

(iv) $\{g \ast f \mid g \in L^1(G), \|g\| < 1\}$ is relatively weakly compact in $L^\infty(G)$.

**Proof.** Let us first show that (i) implies (ii). We denote $A = C^*(G)^{**}$, $A^* = C^*(G)^*$, identify $A^*$ with its canonical image in $A^*$, and write $ag(x) = g(ax)$ for $a, x \in A$, $g \in A^*$. Since $A$ is a C*-algebra, the two Arens products in $A^{**}$ coincide [2, p. 869]. From Theorem 4.2 in [12, p. 91] we see therefore (taking $E = \{x \in A \mid \|x\| < 1\}$ and $\mathfrak{F} = A^*|E$ in the notation of [12]) that for all $g \in A^*$, $\{ag \mid a \in A, \|a\| < 1\}$ is a relatively weakly compact subset of $A^*$. Alternatively, one may apply Corollary II.9 in [1, p. 293] which shows that an arbitrary bounded linear map from a C*-algebra to $A^*$ is a weakly compact operator. If $g \in A^*$, $\{ag \mid a \in A, \|a\| < 1\}$ is a relatively weakly (i.e. $\sigma(A^*, A)$) compact subset of $A^*$, because the multiplication in $A$ is separately continuous in the weak operator topology, which coincides with $\sigma(A, A^*)$ on $A \subset L(H, \omega))$ [3, p. 237]. Let $T : B(G) \to A^*$ be as in §2. Assume now $f \in B(G)$. We show that $\omega(\mu)(Tf) = T(\hat{\mu} \ast f)$ (note that $\hat{\mu} \ast f \in B(G)$ [8, p. 198]; this also follows from the argument below). Since $\omega(\mu)(Tf) \in A^*$,\n
$$\omega(\mu)(Tf) = Th$$

for some $h \in B(G)$. Thus we have (l.a.e.)

$$h(x) = \langle \omega(\mu)(Tf), \omega(\delta_x) \rangle = \langle Tf, \omega(\mu \ast \delta_x) \rangle = \int f d(\mu \ast \delta_x) = \int f(sx) \, d\mu(s) = \int f(s^{-1}x) \, d\hat{\mu}(s) = \hat{\mu} \ast f(x).$$

In [8, p. 197] it is observed that $B(G)$ is invariant under complex conjugation, and $\|Tu\| = \|T\bar{u}\|$ for all $u \in B(G)$. Therefore
\[ \| \tilde{\mu} \|^* = \| (\tilde{\mu})^* \|^* = \| \tilde{\mu} \|^* = \sup \left\{ \left| \int \tilde{u}(x) \, d\mu(x) \right| \mid u \in B(G), \| Tu \| < 1 \right\} \]
\[ = \sup \left\{ \left| \int u(x) \, d\mu(x) \right| \mid u \in B(G), \| Tu \| < 1 \right\} = \| \mu \|^*, \quad \mu \in M(G). \]

After these preparations it is clear that since the linear map \( S: A_\ast \rightarrow L^\infty(G) \)
for which \( S(Tu) = u, u \in B(G), \) is norm decreasing, hence weakly continuous,
the set
\[
\{ \mu \ast f \mid \mu \in M(G), \| \mu \|^* < 1 \} = \{ \tilde{\mu} \ast f \mid \mu \in M(G), \| \mu \|^* < 1 \}
\]
is relatively weakly compact in \( L^\infty(G) \). Obviously, (ii) implies both (iii) and (iv).
Suppose now that (iii) holds. In view of the preceding lemma \( f \) may be assumed to be continuous. Let \( G_d \) denote \( G \) equipped with the discrete topology. Then \( M_d(G) \) can be identified with \( M(G_d) = L^1(G_d) \). Composed with the identity map of \( G_d \) onto \( G \), each continuous unitary representation \( \pi \) of \( G \) defines a unitary representation \( \pi_d \) of \( G_d \), and \( \pi(\mu) = \pi_d(\mu) \) for all \( \mu \in M(G_d) \). Thus, if \( \| \mu \|_d \) for \( \mu \in M(G_d) \) denotes the supremum of \( \| \pi(\mu) \| \)
over the unitary representations \( \pi \) of \( G_d \), then \( \| \mu \|_d \geq \| \mu \|^* \). Since \( \| \mu_n - \mu \|^* \rightarrow 0 \) if \( \mu_n \rightarrow \mu \) in total variation norm, we get
\[
\sup \left\{ \left| \int \tilde{f} \, d\mu \right| \mid \mu \in L^1(G_d), \| \mu \|_d < 1 \right\}
\[ 
\leq \sup \left\{ \left| \mu \ast f(e) \right| \mid \mu \in M_{dd}(G), \| \mu \|^* < 1 \right\}. \]
The latter quantity is finite by assumption, because evaluation at \( e \) is a continuous functional on \( C(G) \). Therefore \( \tilde{f} \in B(G_d) [8, \text{p. 191}] \), and so also \( f \in B(G_d) \). As \( f \) is continuous on \( G, f \in B(G) [8, \text{p. 202}] \). Finally, assume (iv). Since \( \| g \|_d \leq \| g \| \), for \( g \in L^1(G) \), Lemma 3.1 shows that \( f \) may be taken to be continuous. As \( g \ast f \in C(G) \) and \( \int \tilde{f}(x) \, g(x) \, dx = g \ast f(e) \) for \( g \in L^1(G) \), we can again apply Proposition 2.1 in [8, \text{p. 191}] and conclude that \( \tilde{f} \in B(G) \). Thus \( \tilde{f} \in B(G), \) i.e. (i) holds.

4. Characterizations of \( B(G) \cap AP(G) \).

**Theorem 4.1.** For a locally compact group \( G \) and \( f \in L^\infty(G) \) the following four conditions are equivalent:

(i) \( f \) is equal l.a.e. to a function in \( B(G) \cap AP(G) \),
(ii) \( \{ \mu \ast f \mid \mu \in M(G), \| \mu \|^* < 1 \} \) is relatively compact in \( L^\infty(G) \),
(iii) \( \{ \mu \ast f \mid \mu \in M_{dd}(G), \| \mu \|^* < 1 \} \) is relatively compact in \( L^\infty(G) \),
(iv) \( \{ g \ast f \mid g \in L^1(G), \| g \|^* < 1 \} \) is relatively compact in \( L^\infty(G) \).

**Proof.** To prove that (i) implies (ii) we may assume that \( f \in B(G) \cap AP(G) \). Let us first treat the special case where \( G \) is compact. We take [5] as our general reference on the representation theory of compact groups. Let \( \hat{G} \)
denote the dual of \( G \), i.e. the set of all equivalence classes of continuous irreducible unitary representations of \( G \), and choose from each class \( \alpha \in \hat{G} \) a
fixed representation $T_\alpha$ of $G$ on an $n_\alpha$-dimensional Hilbert space $H_\alpha$ (as $G$ is compact, $n_\alpha < \infty$). Denote by $L^{\infty}(\hat{G})$ the space of families $\varphi = (\varphi_\alpha)_{\alpha \in \hat{G}}$ where $\varphi_\alpha \in L(H_\alpha)$ and $\|\varphi\|_{\infty} = \sup\{\|\varphi_\alpha\| \mid \alpha \in \hat{G}\} < \infty$ ($\|\varphi_\alpha\|$ is the usual operator norm). Let $L^1(\hat{G})$ consist of all $\varphi \in L^{\infty}(\hat{G})$ satisfying $\|\varphi\|_1 = \sum_\alpha n_\alpha \text{Tr}(\varphi_\alpha^* \varphi_\alpha)^{1/2} < \infty$, and let $C_0(\hat{G})$ be the set of those $\varphi \in L^{\infty}(\hat{G})$ for which the function $\alpha \mapsto \|\varphi_\alpha\|$ vanishes at infinity on the discrete space $\hat{G}$.

Then $C_0(\hat{G})$ and $L^{\infty}(\hat{G})$ are $C^*$-algebras with coordinatewise operations, and $L^1(\hat{G}) = C_0(\hat{G})^*$, $L^{\infty}(\hat{G}) = L^1(\hat{G})^*$, the duality being implemented by $\langle \varphi, \varphi' \rangle = \sum_\alpha n_\alpha \text{Tr}(\varphi_\alpha^* \varphi_\alpha')$ for $\varphi, \varphi' \in L^1(\hat{G})$, $\varphi, \varphi' \in C_0(\hat{G})$ or $\varphi, \varphi' \in L^{\infty}(\hat{G})$ [5, p. 88]. If $\varphi \in L^1(\hat{G})$ and $\varphi \in L^{\infty}(\hat{G})$, then $\varphi \in L^1(\hat{G})$ and $\|\psi\|_1 < \|\psi\|_1$.

Thus the map $M_\psi : L^{\infty}(\hat{G}) \to L^1(\hat{G})$, $M_\psi(\varphi) = \psi \varphi$, is for each $\psi \in L^1(\hat{G})$ a bounded linear operator, and $\|M_\psi\| < \|\psi\|_1$. If $\{\alpha \in \hat{G} \mid \varphi_\alpha \neq 0\}$ is finite, the range of $M_\psi$ is finite dimensional. Since such elements $\psi$ are dense in $L^1(\hat{G})$, it follows that $M_\psi$ is a compact operator for all $\psi \in L^1(\hat{G})$. It is well known that $C^*(G)$ is isometrically $*$-isomorphic to $C_0(\hat{G})$. (In fact, the Fourier transformation [3, p. 316] $\mathcal{F} : L^1(G) \to C_0(\hat{G})$ is an injective $*$-algebra homomorphism, $\mathcal{F}(L^1(G))$ is dense in $C_0(\hat{G})$ [5, p. 90], and $\mathcal{F}(g)^{\infty} = g'$ [3, pp. 40, 254–255].) As the canonical embedding of $C_0(\hat{G})$ into its bidual can be interpreted as the inclusion of $C_0(\hat{G})$ into $L^{\infty}(\hat{G})$, and $M_\varphi \varphi = \varphi^2$ for $\varphi \in L^1(\hat{G})$, $\varphi \in L^{\infty}(\hat{G})$ (where as usual $\langle \varphi, \varphi' \rangle = \langle \varphi, \varphi' \rangle$, $\varphi', \varphi' \in L^{\infty}(\hat{G})$), the set $\{a \varphi \mid a \in C^*(G)^{**}, \|a\| < 1\}$ is therefore relatively compact in $C^*(G)^*$ for all $g \in C^*(G)^*$. In the case of a compact group $G$, the proof that (i) implies (ii) is now easily completed by the same reasoning as the corresponding part in the proof of Theorem 3.2. In the general case, let $\mathcal{G}$ denote the almost periodic compactification of $G$ [3, p. 297], and $\sigma : G \to \mathcal{G}$ the canonical homomorphism. As $f \in B(G) \cap AP(G)$, there is a unique $\tilde{f} \in B(\mathcal{G})$ satisfying $f = \tilde{f} \circ \sigma$ [8, p. 203]. For $\mu \in M(G)$, let $\tilde{\mu} \in M(\mathcal{G})$ be defined by $\langle \tilde{\mu}, g \rangle = \int g \circ \sigma(x) \, d\mu(x), \, g \in C^0(\mathcal{G})$. It is easily verified that $(\tilde{\mu} \ast \tilde{f}) \circ \sigma = \mu \ast f$ ($\in B(G) \subset C(G)$, see the proof of Theorem 3.2). Furthermore, if $\pi : \mathcal{G} \to L(H)\alpha$ is any continuous unitary representation, and $\xi, \eta \in H_\alpha$, then

$$\int (\pi \circ \sigma(x) \xi, \eta) \, d\mu(x) = \int (\pi(y) \xi, \eta) \, d\tilde{\mu}(y),$$

so that $\pi \circ \sigma(\mu) = \pi(\tilde{\mu})$. This shows that $\|\tilde{\mu}\|' < \|\mu\|'$, and so

$$\{ \mu \ast f \mid \mu \in M(G), \|\mu\|' < 1 \} \subset \{ (\tilde{\mu} \ast \tilde{f}) \circ \sigma \mid \mu \in M(G), \|\mu\|' < 1 \}$$

$$\subset \{ (\nu \ast \tilde{f}) \circ \sigma \mid \nu \in M(\mathcal{G}), \|\nu\|' < 1 \}.$$

Recalling the first part of the proof we now see that, since $g \mapsto g \circ \sigma$ is an isometry from $C(\mathcal{G})$ into $C(G)$, even in the general case (i) implies (ii). Clearly, (ii) implies both (iii) and (iv). Finally, if we assume either (iii) or (iv), then $f$ is by Lemma 3.1 equal l.a.e. to a function in $AP(G)$. By Theorem 3.2 this function belongs to $B(G)$, and so (i) holds.

**Remark.** If $G$ is a locally compact Abelian group and $\Gamma$ its dual group, then $\|\mu\|^{\infty} = \|\tilde{\mu}\|_\infty$ for $\mu \in M(G)$, $B(G) = \{ \tilde{f} \nu \in M(\Gamma) \}$ (§2), and $B(G) \cap AP(G) = \{ \tilde{f} \nu \in M_{geo}(\Gamma) \}$ (see e.g. [8, p. 204]). Therefore the theorem in [11, p. 84] and Theorem 4 in [7, p. 453] follow from Theorem 3.2, and Theorem 5 in [7, p. 453] follows from Theorem 4.1.
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