

ON NONISOMORPHIC ANALYTIC SETS

R. DANIEL MAULDIN

ABSTRACT. It is shown that if A is an analytic subset of I , the unit interval, such that $I - A$ is uncountable and does not contain a perfect set, then A is not Borel isomorphic to $I \times A$ or to A^n , $n > 1$, or to U , where U is a universal analytic subset of I^2 . It is also shown that U is not isomorphic to $I \times A$ or to A^n , $n > 1$.

We shall say that two Borel structures (or measurable spaces) (X, Σ) and (Y, \mathcal{Q}) are isomorphic provided there is a one-to-one measurable map of X onto Y whose inverse is also measurable. It can be seen that if X is isomorphic to a subset Z of Y , where $Z \in \mathcal{Q}$, and Z has the restricted Borel structure, and Y is isomorphic to a subset E of X , where $E \in \Sigma$ and E has the restricted Borel structure, then (X, Σ) and (Y, \mathcal{Q}) are isomorphic. It is well known that if B_1 and B_2 are Borel subsets of Polish spaces provided with the relative Borel structure, then B_1 and B_2 are isomorphic if and only if they have the same cardinality.

The problem of the number of isomorphism classes of analytic, nonborelian subsets of Polish spaces seems to be unsolved. In [1], A. Maitra and C. Ryll-Nardzewski show that (i) any two universal analytic sets are isomorphic, and (ii) if A is an analytic set whose complement is uncountable and does not contain a perfect set, then A is not in the isomorphic class of the universal analytic sets.

In this note we give some corollaries of the techniques employed in [1], and recount their main argument in Theorem 1.

First, let us set some notation. The unit interval will be denoted by I and A will denote an analytic subset of I whose complement is uncountable and does not contain a perfect set. The existence of such a set is implied by Gödel's Axiom of Constructibility [2]. The n -fold product of A with itself is denoted by A^n . The symbol U will denote a universal analytic subset of $I \times I$. The dyadic rationals are denoted by R_0 . The binary sieve of Lebesgue is used throughout this paper [3, p. 34].

In this note, we show that no two of the following sets are isomorphic: A , $I \times A$, and U . We show that U and A^n , $n \geq 1$, are not isomorphic. Also, A

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and $A^n, n > 1$, are shown to be nonisomorphic.

Results.

THEOREM 1. *If B is an analytic subset of a Polish space S and uncountably many of the constituents of B with respect to some sieve are uncountable, then B is not isomorphic to A .*

PROOF. Let $\{C_\alpha\}_{\alpha < \omega_1}$ be the constituents of $S - B$ [3, p. 499] with respect to some sieve so that uncountably many of the C_α 's are uncountable.

Now, suppose ϕ is a Borel isomorphism of A onto B . We may suppose Z_1 is a Borel subset of $I, Z_1 \supseteq A, Z_2$ is a Borel subset of $S, Z_2 \supseteq B$ and ϕ has been extended to a Borel isomorphism of Z_1 onto Z_2 [3, p. 436].

There is an ordinal $\alpha_0, \alpha_0 < \omega_1$, such that $S - Z_2 \subseteq \bigcup_{\beta < \alpha_0} C_\beta$. Fix $\alpha_1 \geq \alpha_0$ such that C_{α_1} is uncountable. Then $C_{\alpha_1} \subseteq Z_2 - B$ and $\phi^{-1}(C_{\alpha_1})$ is an uncountable Borel set lying in $Z_1 - A$. Contradiction.

COROLLARY 1. *The set A is not isomorphic to $I \times A$.*

PROOF. Let L be a sieve of Borel sets such that A is the set sifted by L . Let $\{C_\alpha\}_{\alpha < \omega_1}$ be the constituents of A' with respect to the sieve L . It is easy to show that $\{I \times C_\alpha\}_{\alpha < \omega_1}$ are the constituents of $I \times A'$ with respect to the sieve L_1 , where L_1 is the map from R_0 into the Borel subsets of $I \times I$, defined by $L_1(r) = I \times L(r)$. The corollary follows from Theorem 1.

COROLLARY 2. *The set A is not isomorphic to $A \times A$.*

PROOF. Let L be a sieve of closed subsets of $I \times I$ which sifts $A \times A$ [3, p. 483]. Suppose there is an ordinal α_0 such that if $\alpha \geq \alpha_0, C_\alpha$ is countable, where $\{C_\alpha\}_{\alpha < \omega_1}$ are the constituents of $I^2 - A^2$ with respect to L .

Then $B = \bigcup_{\alpha \leq \alpha_0} C_\alpha$ is a Borel subset of $I \times I$ and $B \subseteq (I \times I) - (A \times A)$. Thus, $\Pi_2(B \cap (A \times I))$, the projection of $B \cap (A \times I)$ into the second axis, is an analytic subset of I disjoint from A . Thus, $D_2 = \Pi_2(B \cap (A \times I))$ is countable. Similarly, $D_1 = \Pi_1(B \cap (I \times A))$ is countable.

Thus, $K = B - [(D_1 \times I) \cup (I \times D_2)]$ is a Borel set lying in $A' \times A'$. From this it follows that $E = \Pi_1(K)$ is countable.

Let $x \in I - (A \cup D_1 \cup E)$. Then $\{x\} \times (I - D_2)$ is a Borel set lying in $(I \times I) - [(A \times A) \cup B]$.

For some $\gamma_0, \gamma_0 < \omega_1, \{x\} \times (I - D_2) \subset \bigcup_{\alpha_0 < \alpha < \gamma_0} C_\alpha$. Thus, some C_α , with $\alpha_0 < \alpha < \gamma_0$ must be uncountable. Therefore, uncountably many of the constituents of $A \times A$ are uncountable and Corollary 2 now follows from Theorem 1. This answers a problem stated by S. Ulam on p. 10 of his problem book [4] assuming the existence of such a set as A . This type of problem is also raised by J.P.R. Christensen in his book [7, pp. 46-47] and answers some of the problems posed there by him.

We shall now demonstrate a property which the sets $A^n, n > 1$, have with respect to any Polish space in which they are embedded, which prevents them from being isomorphic to U .

THEOREM 2. *For each $n > 1$, there is a family \mathcal{G}_n of \aleph_1 subsets of I^n filling up $I^n - A^n$ such that (1) every analytic subset of $I^n - A^n$ is covered by a countable subfamily of \mathcal{G}_n and (2) each set in \mathcal{G}_n is an F_σ set.*

PROOF. The proof proceeds by induction.

First, let us show this for $n = 2$: Again let L be a sieve of Borel sets such that A is the set sifted by L . Let $\{C_\alpha\}_{\alpha < \omega_1}$ be the constituents of A' with respect to the sieve L . The sets C_α are countable.

Let $\mathcal{G}_2 = \{(C_\alpha \times I) \cup (I \times C_\alpha) : \alpha < \omega_1\}$; \mathcal{G}_2 is a family of \aleph_1 subsets of $I^2 - A^2$ and clearly each set in \mathcal{G} is an F_σ set. If B is an analytic set sitting in $I^2 - A^2$, then there is an ordinal α_0 such that $D_1 \cup D_2 \subset \bigcup_{\beta < \alpha_0} C_\beta$, where $D_1 = \Pi_1(B \cap (I \times A))$ and $D_2 = \Pi_2(B \cap (A \times I))$. Thus, $K = B - [(D_1 \times I) \cup (I \times D_2)]$ is an analytic set lying in $A' \times A'$. Therefore, there is an ordinal α_1 , $\alpha_0 < \alpha_1 < \omega_1$, so that $\Pi_i(K) \subset \bigcup_{\beta < \alpha_1} C_\beta$, $i = 1, 2$. It follows that B is covered by $\{(C_\alpha \times I) \cup (I \times C_\alpha) : \alpha < \alpha_1\}$.

Now, let us suppose the existence of such a family \mathcal{G}_n has been shown for the positive integer n , $n > 1$.

Let $\mathcal{G}_{n+1} = \{(I^n \times C_\alpha) \cup (K \times I) : \alpha < \omega_1 \text{ and } K \in \mathcal{G}_n\}$. Clearly, $|\mathcal{G}_{n+1}| = \aleph_1$ and each set in \mathcal{G}_{n+1} is an F_σ set. That \mathcal{G}_{n+1} has the required covering property may be shown by an argument similar to that given for the case $n = 2$.

THEOREM 3. *Suppose D is an analytic subset of a Polish space S such that there is a family of G of \aleph_1 Borel subsets of S filling up $S - D$ such that (1) if E is an analytic subset of S lying in $S - D$, then E is covered by countably many members of G and (2) there is an ordinal α , $\alpha < \omega_1$, such that each member of G is of Borel's additive class α . Then D is not isomorphic to K , if K is an analytic subset of I^2 such that Borel sets of arbitrarily high class appear as vertical sections of K .*

PROOF. Suppose D is isomorphic to K . We may assume that Z_1 is a Borel subset of S with $Z_1 \supset D$, Z_2 is a Borel subset of I^2 with $Z_2 \supset K$ and ϕ is an isomorphism of Z_1 onto Z_2 which takes D onto K . Let γ be an ordinal such that if E is of Borel's additive class α with respect to Z_1 , then $\phi(E)$ is of additive class γ in I^2 . In particular, Z_2 is of additive class γ in I^2 and if $H \in G$, $\phi(H)$ is of additive class γ in I^2 .

Let x be a point of $\Pi_1(K)$ such that the fibre of K over x , K_x , is a Borel subset of I , but not of additive class $\gamma + 2$. Let $T = (Z_2)_x - K_x$. Let $\{H_n\}_{n=1}^\infty$ be a sequence of sets from \mathcal{G} such that $\phi^{-1}(T) \subset \bigcup_{n=1}^\infty H_n$. Then $W = \phi(\bigcup H_n)$ is of additive class γ in I^2 . Since $(I^2 - Z_2)_x$ is of additive class $\gamma + 1$ in I , $W_x \cup (I^2 - Z_2)_x = I - K_x$ is of additive class $\gamma + 1$ in I . Therefore, K_x is of additive class $\gamma + 2$ in I . This contradiction proves the theorem.

COROLLARY 3. *For each n , $n > 1$, the set A^n is not isomorphic with U . Also, the sets $I \times A$ and U are not isomorphic.*

PROOF. That A^n and U are not isomorphic follows immediately from

Theorems 2 and 3 and the fact that all Borel subsets of I appear as vertical sections of U . That $I \times A$ and U are not isomorphic follows from Theorem 3 and the fact that there is a collection G of \aleph_1 subsets of $I^2 - (I \times A)$ satisfying the hypothesis of Theorem 3. (This is indicated in the proof of Corollary 1.)

Finally, let us note that the methods used in this note to show that two analytic sets are not isomorphic involve an analysis of the structure of the complements of these sets in any Polish space in which they are embedded. There are, however, two intrinsic invariants of the isomorphism classes of analytic sets. They are put forth in the following theorems.

THEOREM 4. *The analytic sets A_1 and A_2 are isomorphic if and only if the Banach spaces consisting of the bounded Baire functions over each set provided with the least upper bound norm are isometrically isomorphic.*

An argument for Theorem 5 is given in Theorem 2 of [5].

THEOREM 5. *The analytic sets A_1 and A_2 are isomorphic if and only if the semigroups of analytic relations over these sets are isomorphic as semigroups.*

This is a corollary of Theorem 2.2 of [6].

Finally, I would like to close with the

Question. If $n \neq m$, is it true that A^n is not isomorphic to A^m ?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611