EFFECTIVENESS AND VAUGHT’S GAP \( \omega \) TWO-CARDINAL THEOREM

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ABSTRACT. We consider functions \( f \) with the property that whenever \( \sigma \) is a sentence in \( L_{\omega_1} \), then \( f(\sigma) < \omega \), and if \( \sigma \) has a gap \( > f(\sigma) \) model, then \( \sigma \) admits all types. A question of Barwise is answered by showing that no such \( f \) is recursive, and that the least such \( f \) is not co-r.e.

Barwise proves in [1] an effective version of Vaught’s gap \( \omega \) two-cardinal theorem [7] for a class of sentences which have a certain syntactic form. He then asks (Problem 2.12 of [1]) the following question concerning the possibility of extending this result to the set of all sentences: What can be said about the effectiveness of a function \( f \) (and, in particular, the least \( f \)) such that for any sentence \( \sigma \), if \( \sigma \) has a gap \( > f(\sigma) \) model, then \( \sigma \) admits all types? Note that we have reformulated Barwise’s question in a way that best fits our answer.

First, some notation and definitions. Let \( \mathcal{L} \) be a sufficiently rich, recursive, first-order language which has among its symbols a distinguished unary predicate symbol \( U \). We will deal throughout with this language \( \mathcal{L} \). We let \( \Sigma \) denote the set of all \( \mathcal{L} \)-sentences. A type is a pair \((k, \lambda)\) of infinite cardinals such that \( k > \lambda \), and we say that the \( \mathcal{L} \)-structure \( \mathcal{A} = (A, U, \ldots) \) has type \((k, \lambda)\) iff \( \text{card}(A) = k \) and \( \text{card}(U) = \lambda \). A sentence \( \sigma \) admits the type \((k, \lambda)\) if it has a model of type \((k, \lambda)\). For each \( n < \omega \) and infinite cardinal \( \lambda \), we define \( \tau_n(\lambda) \) inductively: \( \tau_0(\lambda) = \lambda \), and \( \tau_{n+1}(\lambda) = 2^\kappa \) where \( \kappa = \tau_n(\lambda) \). A structure \( \mathcal{A} \) of type \((k, \lambda)\) is a gap \( > n \) structure iff \( k > \tau_n(\lambda) \).

We can now state Vaught’s gap \( \omega \) theorem.

**Vaught’s Theorem.** If \( \sigma \) has a gap \( > n \) model for each \( n > \omega \), then \( \sigma \) admits all types.

Barwise defines, for each \( n < \omega \), a class of formulas which he calls \( \exists \forall^{(n)} \)-mod \( U \) formulas. By induction on \( n \), we define when \( \phi \) is a \( \forall^{(n)} \)-mod \( U \) formula: \( \phi \) is a \( \forall^{(0)} \)-mod \( U \) formula iff it is universal, and \( \phi \) is a \( \forall^{(n+1)} \)-mod \( U \) formula iff it is in the form \( \forall x (\exists y_1 \in U) (\exists y_2 \in U) \ldots (\exists y_m \in U) \phi_1 \), where \( \phi_1 \) is a \( \forall^{(n)} \)-mod \( U \) formula. Then, a formula is a \( \exists \forall^{(n)} \)-mod \( U \) formula if it is obtained from a \( \forall^{(n)} \)-mod \( U \) formula by means of existential quantification.
Barwise's Theorem. If $\sigma$ is a $\exists \forall^{(n+1)}$-mod $U$ sentence which has a gap $> n$ model, then $\sigma$ admits all types.

Barwise's Theorem is optimal in the sense that for each $n < \omega$ there is a $\forall^{(n+1)}$-mod $U$ sentence $\sigma$ which admits each type $(\exists_n(\lambda), \lambda)$ but has no gap $> n$ model. (See Example 2.7 of [1].) It will be shown in this paper that there are certain limitations in extending Barwise's Theorem so as to include all sentences.

Theorem 1. Suppose $f: \Sigma \to \omega$ is such that each sentence $\sigma \in \Sigma$ has an infinite model whenever it has one of cardinality $> f(\sigma)$. Then $f$ is not recursive.

Proof. Let $P, Q$ be unary predicate symbols in $\mathcal{L}$. Let $X \subseteq \omega$ be an r.e., nonrecursive set, and let $g: \omega \to X$ be a recursive bijection. It is known that there is a sentence $\sigma$ such that for each $n < \omega$, there is a unique $\mathcal{A} \models \sigma$ such that $\text{card}(P^\mathcal{A}) = n$; furthermore, this $\mathcal{A}$ is finite and $\text{card}(Q^\mathcal{A}) = g(n)$. Let us refer to this $\mathcal{A}$ as $\mathcal{A}_n$ and let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the language of $\sigma$.

For each $n < \omega$ define a structure $\mathcal{B}_n$ as follows. Without loss of generality, assume that $\{0, \ldots, n\}, A_0, \ldots, A_n$ are pairwise disjoint and let $D$ be their union. Let $E$ be the set of all one-one functions $e: D \to D$, and furthermore assume that $F \cap D = 0$. Let $\preceq$ and $I$ be binary relation symbols and $F$ a ternary relation symbol in $\mathcal{L} - \mathcal{L}_0$. Let $\mathcal{L}_1 = \mathcal{L}_0 \cup \{\preceq, I, F\}$. Now define $\mathcal{B}_n$ to be the $\mathcal{L}_1$-structure where

\[
\begin{align*}
B_n &= E \cup D, \\
R^{\mathcal{B}_n} &= \bigcup \{R^\mathcal{A}_i: i \leq n\} \text{ for } R \in \mathcal{L}_0, \\
\preceq^{\mathcal{B}_n} &= \{\langle i, j \rangle: i \leq j \leq n\}, \\
I^{\mathcal{B}_n} &= \{\langle i, a \rangle: i \leq n \text{ and } a \in A_i\}, \\
F^{\mathcal{B}_n} &= \{\langle e, x, y \rangle: e \in E, x, y \in D \text{ and } y = e(x)\}.
\end{align*}
\]

It is easy to see that there is a sentence $\phi$ whose finite models are just the isomorphs of the $\mathcal{B}_n$'s. Also, if $\mathcal{B} \models \phi$ is infinite, then each $\mathcal{B}_n$ is embeddable in $\mathcal{B}$.

For each $m < \omega$ let $\phi_m$ be the sentence $\phi \land \forall x \rightarrow \exists^m y (I(x, y) \land Q(y))$, where $\exists^m$ is the quantifier "there are exactly $m$". Whenever $m, n < \omega$, then

\[\mathcal{B}_n \models \phi_m \iff m \not\in \{g(0), \ldots, g(n)\}.\]

Thus, $\phi_m$ has an infinite model iff $m \not\in X$.

Now suppose $f: \Sigma \to \omega$ is as described in the theorem, and, in addition, is recursive. To see if $\phi_m$ has an infinite model, let $k = f(\phi_m)$. Certainly $\text{card}(B_k) > k$, so that if $\mathcal{B}_k \models \phi_m$, then $\phi_m$ has an infinite model. Conversely, if $\phi_m$ has an infinite model, then $m \not\in X$ so that $\mathcal{B}_k \not\models \phi_m$. Thus $\mathcal{B}_k \models \phi_m$ iff $m \not\in X$, and this contradicts the fact that $X$ is not recursive. \qed

Remark 2. The least such $f$ as described in Theorem 1 is, however, $\Pi^0_1$. For $f$ can be defined by $f(\sigma) = \mu n (\text{if } \sigma \text{ has a finite model of cardinality } \geq n, \text{ then it has an infinite model}).$

Corollary 3. Suppose $f: \Sigma \to \omega$ is such that each sentence $\sigma \in \Sigma$ which has...
a gap $> f(\sigma)$ model admits all types. Then $f$ is not recursive.

Proof. From standard examples of two-cardinal models (e.g. Example 2.13 of [1]), it is easy to find a recursive function $g: \Sigma \to \Sigma$ such that $g(\sigma)$ has a gap $> n$ model iff $\sigma$ has a model of cardinality $> n$. Now, if $f: \Sigma \to \omega$ were a recursive function as in the corollary, then the function $f \circ g$ would contradict Theorem 1. □

Theorem 4. There is a $\Pi_1^0$ function $f: \Sigma \to \omega$ such that each $\sigma \in \Sigma$ which has a gap $> f(\sigma)$ model admits all types.

Proof. Any proof of Vaught's Theorem falls naturally into two parts. First of all it is shown that there is a recursive set $\Gamma$ of sentences (in a language larger than $\mathcal{E}$) such that $\sigma$ admits all types whenever $\{\sigma\} \cup \Gamma$ is consistent. Secondly, there is a recursive sequence $\Gamma_0, \Gamma_1, \ldots$ such that $\Gamma = \bigcup \{\Gamma_n: n < \omega\}$ and such that $\{\sigma\} \cup \Gamma_n$ is consistent whenever $\sigma$ has a gap $> n$ model.

Notice that the set $\{\sigma \in \Sigma: \sigma$ admits all types$\}$ is $\Pi_1^0$ since this is just the set of $\sigma$ for which $\{\sigma\} \cup \Gamma$ is consistent. Let $R(x, y, z)$ be a recursive predicate such that $\exists x R(x, \sigma, n)$ iff $\{\sigma\} \cup \Gamma_n$ is inconsistent. Now define $f: \Sigma \to \omega$ so that

$$f(\sigma) = n \leftrightarrow [n = 0 \land (\sigma$ admits all types$)] \lor
\[(\exists x < n)(\exists m < n)R(x, \sigma, m) \land
(\forall k < n)(\forall x < k)(\forall m < k) \rightarrow R(x, \sigma, n)].$$

Clearly $f$ is $\Pi_1^0$. To see that $f$ works, suppose that $\sigma$ has a gap $> n$ model, where $n = f(\sigma)$. We claim that $n = 0$, so that $\sigma$ admits all types. For, if not, there are $x < n$ and $m < n$ such that $R(x, \sigma, m)$, and this means that $\{\sigma\} \cup \Gamma_m$ is inconsistent. But this contradicts the fact that $\sigma$ has a gap $> m$ model. □

Let us say that $X \subseteq \omega$ is $\Sigma_{1,2}^0$ if there are r.e. sets $A$ and $B$ such that $X = A - B$.

Theorem 5. Let $f: \Sigma \to \omega$ be the least function such that each $\sigma \in \Sigma$ which has a gap $> f(\sigma)$ model admits all types. Then $f$ is not $\Pi_1^0$, but (assuming $V = L$) is $\Sigma_{1,2}^0$.

Proof. First, by way of contradiction, assume that $f$ is $\Pi_1^0$. Let $X = \{\sigma \in \Sigma: f(\sigma) = 1\}$; thus, $X$ consists of those $\sigma$ which have a gap $> 0$ model but no gap $> 1$ model. It is easy to see that for each sentence $\phi$, one can effectively get a sentence $\phi_\phi$ such that: $\phi_\phi$ has a gap $> n$ model iff $\phi$ has a model $\mathfrak{a}$ for which $\text{card}(U^\mathfrak{a}) \geq n$. Thus $\sigma_\phi \in X$ iff both

1. $\phi$ is consistent;
2. $\vDash \phi \rightarrow \exists x U(x)$.

Let $A$ be the set of $\phi$ satisfying (1), and let $B$ be the set of $\phi$ satisfying (2). Notice that $B$ is not $\Pi_1^0$ since, if $\phi$ does not involve $U$, then $\phi \in B$ iff $\phi$ is inconsistent.

Now for each $\phi$ let $\phi' = \phi \lor \neg \exists x U(x)$. Then $\phi \in B$ iff $\phi' \in A \cap B$. Thus $A \cap B$ is not $\Pi_1^0$ so that neither is $f$.

It follows from Jensen's proof [2] of the gap $n$ conjecture under the assumption of $V = L$, that there is a recursive sequence $\Gamma_0, \Gamma_1, \ldots$ of sets of
sentences such that $\sigma$ has a gap $> n$ model iff $\{\sigma\} \cup \Gamma_n$ is consistent. Then $f$ is easily seen to be $\Sigma^0_{1,2}$. □

There are natural and obvious analogues of Corollary 3, Theorem 4 and the first part of Theorem 5 that correspond to other transfer theorems. In particular, we refer to Theorem 1 of [4], Theorem 2.17(A) of [5], Theorems 4.1 and 4.4 of [3], and Theorem 1 of [6].

REFERENCES