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EFFECTIVENESS AND VAUGHT'S GAP ω TWO-CARDINAL THEOREM

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ABSTRACT. We consider functions f with the property that whenever σ is a sentence in $L_{\omega\omega}$, then $f(\sigma) < \omega$, and if σ has a gap $> f(\sigma)$ model, then σ admits all types. A question of Barwise is answered by showing that no such f is recursive, and that the least such f is not co-r.e.

Barwise proves in [1] an effective version of Vaught's gap ω two-cardinal theorem [7] for a class of sentences which have a certain syntactic form. He then asks (Problem 2.12 of [1]) the following question concerning the possibility of extending this result to the set of all sentences: What can be said about the effectiveness of a function f (and, in particular, the least f) such that for any sentence σ, if σ has a gap $> f(\sigma)$ model, then σ admits all types? Note that we have reformulated Barwise's question in a way that best fits our answer.

First, some notation and definitions. Let $\mathfrak{L}$ be a sufficiently rich, recursive, first-order language which has among its symbols a distinguished unary predicate symbol U. We will deal throughout with this language $\mathfrak{L}$. We let Σ denote the set of all $\mathfrak{L}$-sentences. A *type* is a pair (κ, λ) of infinite cardinals such that $\kappa \geqslant \lambda$, and we say that the $\mathfrak{L}$-structure $\mathfrak{A} = (A, U, \dots)$ has type (κ, λ) iff card $(A) = \kappa$ and card $(U) = \lambda$. A sentence σ *admits* the type (κ, λ) if it has a model of type (κ, λ). For each $n < \omega$ and infinite cardinal λ, we define $\beth_n(\lambda)$ inductively: $\beth_0(\lambda) = \lambda$, and $\beth_{n+1}(\lambda) = 2^\kappa$ where $\kappa = \beth_n(\lambda)$. A structure $\mathfrak{A}$ of type (κ, λ) is a *gap* $> n$ structure iff $\kappa > \beth_n(\lambda)$.

We can now state Vaught's gap ω theorem.

VAUGHT'S THEOREM. *If σ has a gap $> n$ model for each $n > \omega$, then σ admits all types.*

Barwise defines, for each $n < \omega$, a class of formulas which he calls $\vec{\exists}\forall^{(n)}$-mod U *formulas*. By induction on n, we define when ϕ is a $\forall^{(n)}$-mod U formula: ϕ is a $\forall^{(0)}$-mod U formula iff it is universal, and ϕ is a $\forall^{(n+1)}$-mod U formula iff it is in the form $\forall x(\exists y_1 \in U)(\exists y_2 \in U)\dots(\exists y_m \in U)\phi_1$, where ϕ_1 is a $\forall^{(n)}$-mod U formula. Then, a formula is a $\vec{\exists}\forall^{(n)}$-mod U formula if it is obtained from a $\forall^{(n)}$-mod U formula by means of existential quantification.

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BARWISE'S THEOREM. *If σ is a $\vec{\exists}\forall^{(n+1)}$-mod U sentence which has a gap $> n$ model, then σ admits all types.*

Barwise's Theorem is optimal in the sense that for each $n < \omega$ there is a $\forall^{(n+1)}$-mod U sentence σ which admits each type $(\beth_n(\lambda), \lambda)$ but has no gap $>$ n model. (See Example 2.7 of [1].) It will be shown in this paper that there are certain limitations in extending Barwise'sTheorem so as to include all sentenc-es.

THEOREM 1. *Suppose f: $\Sigma \to \omega$ is such that each sentence $\sigma \in \Sigma$ has an infinite model whenever it has one of cardinality $> f(\sigma)$. Then f is not recursive.*

PROOF. Let P, Q be unary predicate symbols in $\mathfrak{L}$. Let $X \subseteq \omega$ be an r.e., nonrecursive set, and let g: $\omega \to X$ be a recursive bijection. It is known that there is a sentence σ such that for each $n < \omega$, there is a unique $\mathfrak{A} \models \sigma$ such that card $(P^{\mathfrak{A}}) = n$; furthermore, this $\mathfrak{A}$ is finite and card $(Q^{\mathfrak{A}}) = g(n)$. Let us refer to this $\mathfrak{A}$ as $\mathfrak{A}_n$, and let $\mathfrak{L}_0 \subseteq \mathfrak{L}$ be the language of σ.

For each $n < \omega$ define a structure $\mathfrak{B}_n$ as follows. Without loss of generality, assume that $\{0, \ldots, n\}$, $A_0, \ldots, A_n$ are pairwise disjoint and let D be their union. Let E be the set of all one-one functions e: $D \to D$, and furthermore assume that $E \cap D = 0$. Let $\leqslant$ and I be binary relation symbols and F a ternary relation symbol in $\mathfrak{L} - \mathfrak{L}_0$. Let $\mathfrak{L}_1 = \mathfrak{L}_0 \cup \{\leqslant, I, F\}$. Now define $\mathfrak{B}_n$ to be the $\mathfrak{L}_1$-structure where

$$\begin{aligned}
B_n &= E \cup D,\\
R^{\mathfrak{B}_n} &= \bigcup\{R^{\mathfrak{A}_i}: i \leqslant n\} \text{ for } R \in \mathfrak{L}_0,\\
\leqslant^{\mathfrak{B}_n} &= \{\langle i,j\rangle: i \leqslant j \leqslant n\},\\
I^{\mathfrak{B}_n} &= \{\langle i,a\rangle: i \leqslant n \text{ and } a \in A_i\},\\
F^{\mathfrak{B}_n} &= \{\langle e,x,y\rangle: e \in E, x,y \in D \text{ and } y = e(x)\}.
\end{aligned}$$

It is easy to see that there is a sentence ϕ whose finite models are just the isomorphs of the $\mathfrak{B}_n$'s. Also, if $\mathfrak{B} \models \phi$ is infinite, then each $\mathfrak{B}_n$ is embeddable in $\mathfrak{B}$.

For each $m < \omega$ let ϕ_m be the sentence $\phi \wedge \forall x \neg \exists^m y(I(x,y) \wedge Q(y))$, where $\exists^m$ is the quantifier "there are exactly m". Whenever m, $n < \omega$, then

$$\mathfrak{B}_n \models \phi_m \quad \text{iff } m \notin \{g(0), \ldots, g(n)\}.$$

Thus, ϕ_m has an infinite model iff $m \notin X$.

Now suppose f: $\Sigma \to \omega$ is as described in the theorem, and, in addition, is recursive. To see if ϕ_m has an infinite model, let $k = f(\phi_m)$. Certainly card $(B_k) > k$, so that if $\mathfrak{B}_k \models \varphi_m$, then ϕ_m has an infinite model. Conversely, if φ_m has an infinite model, then $m \notin X$ so that $\mathfrak{B}_k \models \phi_m$. Thus $\mathfrak{B}_k \models \phi_m$ iff $m \notin X$, and this contradicts the fact that X is not recursive. □

REMARK 2. The least such f as described in Theorem 1 is, however, Π_1^0. For f can be defined by $f(\sigma) = \mu n$ (if σ has a finite model of cardinality $> n$, then it has an infinite model).

COROLLARY 3. *Suppose f: $\Sigma \to \omega$ is such that each sentence $\sigma \in \Sigma$ which has* *a gap $> f(\sigma)$ model admits all types. Then f is not recursive.*

PROOF. From standard examples of two-cardinal models (e.g. Example 2.13 of [1]), it is easy to find a recursive function $g: \Sigma \to \Sigma$ such that $g(\sigma)$ has a gap $> n$ model iff σ has a model of cardinality $> n$. Now, if $f: \Sigma \to \omega$ were a recursive function as in the corollary, then the function $f \circ g$ would contradict Theorem 1. □

THEOREM 4. *There is a Π^0_1 function $f: \Sigma \to \omega$ such that each $\sigma \in \Sigma$ which has a gap $> f(\sigma)$ model admits all types.*

PROOF. Any proof of Vaught's Theorem falls naturally into two parts. First of all it is shown that there is a recursive set Γ of sentences (in a language larger than $\mathfrak{L}$) such that σ admits all types whenever $\{\sigma\} \cup \Gamma$ is consistent. Secondly, there is a recursive sequence $\Gamma_0, \Gamma_1, \ldots$ such that $\Gamma = \bigcup\{\Gamma_n: n < \omega\}$ and such that $\{\sigma\} \cup \Gamma_n$ is consistent whenever σ has a gap $> n$ model.

Notice that the set $\{\sigma \in \Sigma: \sigma \text{ admits all types}\}$ is Π^0_1 since this is just the set of σ for which $\{\sigma\} \cup \Gamma$ is consistent. Let $R(x,y,z)$ be a recursive predicate such that $\exists x R(x,\sigma,n)$ iff $\{\sigma\} \cup \Gamma_n$ is inconsistent. Now define $f: \Sigma \to \Sigma$ so that

$$\begin{aligned} f(\sigma) = n \leftrightarrow {} & [n = 0 \wedge (\sigma \text{ admits all types})] \vee \\ & [(\exists x < n)(\exists m < n) R(x,\sigma,m) \wedge \\ & \quad (\forall k < n)(\forall x < k)(\forall m < k) \neg R(x,\sigma,n)]. \end{aligned}$$

Clearly f is Π^0_1. To see that f works, suppose that σ has a gap $> n$ model, where $n = f(\sigma)$. We claim that $n = 0$, so that σ admits all types. For, if not, there are $x < n$ and $m < n$ such that $R(x,\sigma,m)$, and this means that $\{\sigma\} \cup \Gamma_m$ is inconsistent. But this contradicts the fact that σ has a gap $> m$ model. □

Let us say that $X \subseteq \omega$ is $\Sigma^0_{1,2}$ if there are r.e. sets A and B such that $X = A - B$.

THEOREM 5. *Let $f: \Sigma \to \omega$ be the least function such that each $\sigma \in \Sigma$ which has a gap $> f(\sigma)$ model admits all types. Then f is not Π^0_1, but (assuming $V = L$) is $\Sigma^0_{1,2}$.*

PROOF. First, by way of contradiction, assume that f is Π^0_1. Let $X = \{\sigma \in \Sigma: f(\sigma) = 1\}$; thus, X consists of those σ which have a gap > 0 model but no gap > 1 model. It is easy to see that for each sentence φ, one can effectively get a sentence σ_ϕ such that: σ_ϕ has a gap $> n$ model iff ϕ has a model $\mathfrak{A}$ for which card $(U^{\mathfrak{A}}) \geqslant n$. Thus $\sigma_\phi \in X$ iff both

(1) ϕ is consistent;

(2) $\models \phi \to \neg \exists x U(x)$.

Let A be the set of ϕ satisfying (1), and let B be the set of ϕ satisfying (2). Notice that B is not Π^0_1 since, if ϕ does not involve U, then $\phi \in B$ iff ϕ is inconsistent.

Now for each ϕ let $\phi' = \phi \vee \neg \exists x U(x)$. Then $\phi \in B$ iff $\phi' \in A \cap B$. Thus $A \cap B$ is not Π^0_1 so that neither is f.

It follows from Jensen's proof [2] of the gap n conjecture under the assumption of $V = L$, that there is a recursive sequence $\Gamma_0, \Gamma_1, \ldots$ of sets of sentences such that σ has a gap $> n$ model iff $\{\sigma\} \cup \Gamma_n$ is consistent. Then f is easily seen to be $\Sigma^0_{1,2}$. $\square$

There are natural and obvious analogues of Corollary 3, Theorem 4 and the first part of Theorem 5 that correspond to other transfer theorems. In particular, we refer to Theorem 1 of [4], Theorem 2.17(A) of [5], Theorems 4.1 and 4.4 of [3], and Theorem 1 of [6].

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