INVERSE AND INJECTIVITY OF PARALLEL RELATIONS INDUCED BY CELLULAR AUTOMATA

TAKEO YAKU

ABSTRACT. Moore and Myhill showed that Garden-of-Eden theorem [2], [3]. A binary relation over the configurations is said to be "parallel" if it is induced by a cellular (tessellation) automaton. Richardson showed the equivalence between a parallel relation (a nondeterministic parallel map) with the quiescent state to be injective and its inverse to be parallel by the Garden-of-Eden theorem plus compactness of product topology [4]. This paper deals with the inverse and the injectivity when a cellular automaton is given that induces a parallel relation. We give an equivalent condition, concerning only the local map, for the inverse of a parallel relation to be parallel. Furthermore we show an equivalent condition, concerning only the local map, for the injectivity of a parallel map. Consequently, we note that these two conditions are represented by semirecursive predicates.

1. Introduction. A cellular automaton—also known as a tessellation structure—is a model of an array of uniformly connected identical finite automata arranged in a $d$-dimensional Euclidean space divided into square cells, where $d$ is called the dimension. The cellular automaton is denoted by $M = (V, Z^d, X, f)$, where (i) $V$ is the state set of each finite automaton. (ii) $Z$ denoted the integers. (iii) $X$ is a distinct $n$-tuple $(x_1, x_2, \ldots, x_n)$ from $Z^d$, called the neighbourhood index, where $n$ is a positive integer. We will always assume that $x_1 = 0$ (0 denotes the $p$-tuple of 0's). $X$ denotes the locations of the finite automata which are connected to each finite automaton. (iv) $f \subseteq V^n \times V$ is the state function of each finite automaton called the local relation. We often represent a binary relation $R \subseteq X \times Y$ by a nondeterministic mapping of a subset of $X$ to $2^Y$. A totally defined or a deterministic relation denotes a totally defined or a deterministic mapping, respectively.

A configuration is a mapping $Z^d \rightarrow V$, which is an assignment of states into the array. Now, the parallel relation $R$ (over the configurations) induced by $M$ is defined as follows: For configurations $c$ and $d$,

$$(c, d) \in R \iff f(c(i + x_1), c(i + x_2), \ldots, c(i + x_n)) \supseteq d(i) \quad \forall i \in Z^d.$$ 

A binary relation over the configurations is said to be parallel if it is induced by some cellular automaton. A parallel relation $R$ is called a parallel map if $R$ is deterministic, that is, the local relation is deterministic.
A cellular automaton $M = (V, Z^d, X, f)$ is with the quiescent state if there is a state $v_q$ in $V$ such that $f(v_q, \ldots, v_q) = \{v_q\}$. The parallel relation $R$ induced by $M$ is with the quiescent state if $M$ is with the quiescent state, i.e., $c_q R d$ iff $d = c_q$ for the quiescent configuration $c_q$. A configuration $c$ is with finite support provided that the set $\{i \in Z; c(i) \neq v_q\}$ is finite.

$\overline{X}(i)$ denotes the set $\{(i + x_1), (i + x_2), \ldots, (i + x_n)\}$ and $\overline{X}(A)$ denotes $\bigcup_{i \in A} \overline{X}(i)$. A pattern is a restriction of a configuration to a finite set. The parallel relation $R_p$ over the patterns is defined by: For patterns $p$ and $q$, $(p, q) \in R_p$ iff $\dom p = \overline{X}(\dom q)$ and

$$f(p(i + x_1), p(i + x_2), \ldots, p(i + x_n)) \ni q(i) \ \forall i \in \dom q.$$  

Garden-of-Eden Theorem (Moore [2] and Myhill [3]). A totally defined parallel map $R$ with the quiescent state is surjective if and only if $R$ is injective restricted to configurations with finite support.

Richardson combined the theorem above and compactness of product topology [4], and gave the following theorem.

Theorem A (Richardson [4]). A totally defined parallel map $R$ with the quiescent state is injective if and only if the inverse of $R$ is a totally defined parallel map with the quiescent state.

2. Results.

Definition. Let $R_p$ be a parallel relation over the patterns induced by $M = (V, Z^d, X, f)$. With respect to a finite set $A (0 \in A)$ in $Z^d$, $R_p$ is said to be $A$-independent if for any patterns $p$, $p'$ and $q$ such that $\dom q = A$, $p R_p q$, and $p' R_p q$, $r R_p q$ for the pattern $r$ such that $\dom r = \dom p$, $r(0) = p'(0)$ and $r(i) = p(i)$ for $i \neq 0$.

A set $A$ is said to be sufficiently large with respect to $X$ if $\overline{X}(i) \cap \overline{X}(0) = \emptyset$ or $\overline{X}(i) \cap (Z^d - \overline{X}(A)) = \emptyset$ for any $i$ in $Z^d$.

Lemma 1. Let $R^{-1}$ be the inverse of a totally defined parallel relation $R$ induced by $M = (V, Z^d, X, f)$. If $R^{-1}$ is a parallel relation induced by $M' = (V, Z^d, Y, g)$, then $R_p$ is $A$-independent for some sufficiently large finite set $A$ in $Z^d$.

Proof. Let $A$ be a sufficiently large finite set in $Z^d$ such that $\overline{Y}(0) \subseteq A$ and $0 \in A$. Since $R^{-1}$ is parallel, for patterns $p$, $p'$ and $q$, if $\dom q = A$, $p R_p q$ and $p' R_p q$, then $p(0)$ and $p'(0)$ are in

$$g(q(0 + y_1), q(0 + y_2), \ldots, q(0 + y_n)),$$

where $Y = (y_1, y_2, \ldots, y_n)$. Since $f$ is totally defined, then there are patterns $q_1$ and $q'_1$ such that (i) $\overline{Y}(\overline{X}(A)) = \dom q_1 = \dom q'_1$, (ii) $q_1(i) = q'_1(i) = q(i)$ for $i \in A$, and (iii) $q_1(R^{-1}) p$ and $q'_1(R^{-1}) p'$, where $(R^{-1}) p$ is the parallel relation over the patterns induced by $M'$. Thus from (1), $q_1(R^{-1}) p$ for the pattern $r$ such that $\dom r = \dom p$, $r(0) = p'(0)$ and $r(i) = p(i)$ for $i \neq 0$, since $\overline{Y}(0) \subseteq A$. Accordingly, $r R_p q$, and therefore $R_p$ is $A$-independent.

Q.E.D.

In order to prove the converse of Lemma 1, we will show in advance the following
Lemma 2. Let us suppose that \( R \) is the parallel relation induced by \( M = (V, Z^d, X, f) \). Let \( A \) be a finite set in \( Z^d \) which is sufficiently large with respect to \( X \), and \( A' \) be such as \( A' \supseteq A \). If \( R_p \) is \( A \)-independent, then \( R_p \) is \( A' \)-independent.

Proof. For any patterns \( p_1, p_2 \) and \( q' \) such that \( \text{dom } q' = A' \), if \( p_1 R_p q' \) and \( p_2 R_p q' \), then \( p_1 R_p q \) and \( p_2 R_p q \) for patterns \( p_1, p_2 \) and \( q \) such as \( p_1 \subseteq p_1', p_2 \subseteq p_2', q \subseteq q' \), \( \text{dom } q = A \) and \( \text{dom } p_1 = \text{dom } p_2 = \bar{X}(A) \). Accordingly \( r R_p q \) for the pattern \( r \) such that \( \text{dom } r = \text{dom } p \), \( r(0) = p_2(0) \) and \( r(i) = p_1(i) \) for \( i \neq 0 \). Let \( r' \) be a pattern such as (i) \( \text{dom } r' = \bar{X}(A') \), (ii) \( r'(0) = p_2'(0) \), and (iii) \( r'(i) = p_1'(i) \) for \( i \neq 0 \).

We will prove that \( r' R_p q' \). Let \( r'' \) be the restriction of \( r' \) to \( \bar{X}(i) \). For \( i \) such that \( \bar{X}(i) \ni 0 \), we obtain \( r'' R_p q'(i) (q'(i) = q(i)) \), since \( r''(i) = r(i) \) for \( i \in \bar{X}(i) \), \( R \) is \( A \)-independent and \( A \) is sufficiently large. For \( i \) such as \( \bar{X}(i) \ni 0 \), it is obvious that \( p_1'' R_p p_2 q'(i) \), where \( p_1'' \) is the restriction of \( p_1' \) to \( \bar{X}(i) \).

Q.E.D.

The next theorem does not only show that the inverse of a parallel relation is parallel when \( R_p \) is \( A \)-independent, but also shows that we can explicitly give a cellular automaton that induces the inverse, using a cellular automaton defined below.

Let us suppose that \( A = \{a_1, a_2, \ldots, a_m\} (a_1 = 0) \) is a finite set in \( Z^d \) and that \( M = (V, Z^d, X, f) \) is a cellular automaton with \( X = (x_1, x_2, \ldots, x_n) \). Let \((R_p)^{-1}\) be the inverse of the parallel relation \( R_p \) over the patterns induced by \( M \). Let \( Y = (y_1, y_2, \ldots, y_{n'}) \) be such that

\[
(2) \quad \bar{X}(A) \subseteq \bigcap_{1 \leq i \leq n'} \bar{Y}(x_i)
\]

and that \( n' \) is finite. Now, \( M(A) = (V, Z^d, Y, g) \) denotes a cellular automaton defined as:

\[
(3) \quad g(v_1, v_2, \ldots, v_n) \ni v,
\]

if there are patterns \( p \) and \( q \) such that \( p R_p q \), \( p(0) = v \) and \( p(y_j) = v_j \) for any \( i \) \((1 \leq i \leq n')\) where \( \text{dom } p = \bar{X}(Y(0)) \) and \( \text{dom } q = \bar{Y}(0) \).

Theorem 1. Let \( R \) be a parallel relation induced by \( M = (V, Z^d, X, f) \) with the inverse \( R^{-1} \). Let \( A \) be a sufficiently large finite set in \( Z^d \) with respect to \( X \). If the parallel relation \( R_p \) over the patterns is \( A \)-independent, then a cellular automaton \( M(A) \) defined in (3) induces \( R^{-1} \).

Proof. Let \( A \) be such that \( A = \{a_1, a_2, \ldots, a_m\} \). We can assume that \( a_1 = 0 \). Assume that \( M(A) = (V, Z^d, Y, g), Y = (y_1, y_2, \ldots, y_{n'}) \) and \( S \) is the parallel relation induced by \( M(A) \).

We will prove that \( d R^{-1} c \) if and only if \( d S c \) for configurations \( c \) and \( d \). Let us assume that \( d R^{-1} c \). Let \( p \) be the restriction of \( c \) to \( \bar{X}(Y(i)) \) and \( q \) be the restriction of \( d \) to \( \bar{Y}(i) \) with respect to any \( i \) in \( Z^d \). Clearly \( p R_p q \). Hence \( p(i) \in g(q(i + y_1), q(i + y_2), \ldots, q(i + y_{n'})) \) from (3). Thus \( d S c \).

On the other hand, let us assume that \( d S c \). We must prove that \( d(i) \in f(e(i + x_1), e(i + x_2), \ldots, e(i + x_n)) \) for any \( i \) in \( Z^d \), where \( X = (x_1, x_2, \ldots, x_n) \). Fix \( i \in Z^d \).
From (3) there are patterns $p_1, p_2, \ldots, p_n$ for each $1 \leq j \leq n$ such that

$$
\begin{cases}
\text{dom } p_j = X(Y(i + x_j)), \\
p_j(i + x_j) = c(i + x_j), \\
p_j R_p q_j,
\end{cases}
$$

where $X = (x_1, x_2, \ldots, x_n)$ and $q_j$ is the restriction of $d$ to $Y(i + x_j)$. Let $q_j' (1 \leq j \leq n)$ be such that $q_j' = q_1 \cap q_2 \cap \cdots \cap q_j$, where mappings $q_1, q_2, \ldots, q_j$ are considered as sets.

Let $r_1, r_2, \ldots, r_n$ be the patterns as defined below:

$$
\begin{cases}
r_1 = p_1, \\
dom r_{j+1} = dom r_j \cap dom p_{j+1}, \\
r_{j+1}(x) = r_j(x) \quad (x \neq i + x_{j+1}), \\
r_{j+1}(x) = p_{j+1}(x) \quad (x = i + x_{j+1}),
\end{cases}
$$

where $1 \leq j \leq n$. It is clear that $r_j R_p q_j'$. We will prove that if $r_j R_p q_j'$, then $r_{j+1} R_p q_{j+1}'$ for $j (1 \leq j < n)$.

Assume that $r_j R_p q_j'$. Let $r_{j+1}'$ be the restriction of $r_j$ to dom $r_{j+1}$. We have $r_{j+1} R_p q_{j+1}'$, while $A$ is sufficiently large and $\{i + x_{j+1} + a_1, i + x_{j+1} + a_2, \ldots, i + x_{j+1} + a_m\} \subseteq dom r_{j+1}$ from (2). Thus, from Lemma 2, $r_{j+1} R_p q_{j+1}'$. Accordingly, $r_n R_p q_n'$. Since $q_n'(i) = d(i)$, the proof is completed. Q.E.D.

**Theorem 2.** The inverse of a totally defined parallel relation $R$ induced by $M = (V, Z^d, X, f)$ is parallel if and only if the parallel relation $R_p$ over the patterns is $A$-independent for some sufficiently large finite set $A$ with respect to $X$.

The next theorem deals with the injectivity of a parallel map. From Theorems A and 2

**Theorem 3.** A totally defined parallel map $R$ with the quiescent state induced by $M = (V, Z^d, X, f)$ is injective if and only if

(i) the parallel map $R_p$ over the patterns is $A$-independent for some sufficiently large finite set $A$ with respect to $X$, and

(ii) the inverse $R^{-1}$ is totally defined and with the quiescent state.

We note that condition (i) in Theorem 3 is represented by a semirecursive predicate. While, $R^{-1}$ in Theorem 3 is induced by $M(A)$ defined in (3), if $R_p$ is $A$-independent. Then:

**Remark.** The following are represented by semirecursive predicates:

(i) The inverse of the totally defined parallel relation induced by a given cellular automaton is parallel.

(ii) The totally defined parallel map with the quiescent state induced by a given cellular automaton is injective.

**References**


Department of Mathematics, Waseda University, Tokyo 160, Japan

Current address: Department of Mathematical Sciences, Tokai University, Hiratsuka, Kanagawa 259-12, Japan