SOME APPLICATIONS OF THE STONE-WEIERSTRASS THEOREM TO PLANAR RATIONAL APPROXIMATION

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Abstract. The Stone-Weierstrass theorem is used to prove two general results about algebras of continuous functions, and each of these yields a necessary and sufficient condition for the planar function algebras \( R(K) \) and \( C(K) \) to be coincident.

We begin with a result which slightly extends the Stone-Weierstrass theorem.

Theorem 1. (Almost selfadjoint algebras are selfadjoint.) Let \( X \) be a compact Hausdorff space, and let \( f \in C(X) \). Suppose \( m \) and \( n \) are relatively prime positive integers. Let \( A \) be the closed subalgebra of \( C(X) \) generated by \( f^m, f^n, \) and the constant functions. Then \( A = \{ g \circ f : g \in C(f(X)) \} \). In particular, if \( f \) separates the points of \( X \), then \( A = C(X) \).

Proof. Let \( K = f(X) \) and let \( B \) be the closed subalgebra of \( C(K) \) generated by \( z^m, z^n, \) and the constant functions. We claim \( B = C(K) \). By the usual Stone-Weierstrass theorem, it suffices to show that \( z \in B \) and \( \bar{z} \in B \). Now \( z^m \in B \) and \( z^n \in B \) imply \( (z^m) (z^n)^m \in B \), or \( |z|^{2mn} \in B \). Since the function \( h(u) = u^{1/2mn} \) is uniformly approximable by polynomials on the interval \( [0, \|z\|_K^{2mn}] \) and \( B \) is closed, it follows that \( |z| \in B \). Since \( (m, n) = 1 \), we can find integers \( a \) and \( b \) with \( am + bn = 1 \). Without loss of generality, we can assume that \( a \geq 0, b \leq 0 \). Then on \( K - \{0\} \) we have

\[
z = z^{am+bn} = (z^m)^a/(z^n)^{-b} = (z^m)^a(z^n)^{-b}/|z|^{-2bn}
\]

and

\[
\bar{z} = z^n/z^{n-1} = \bar{z}^n z^{n-1}/|z|^{2n-2}.
\]

For each \( \epsilon > 0 \), let

\[
h_\epsilon(u) = \begin{cases} 1/\epsilon^{-2bn} & \text{on } [0, \epsilon], \\ 1/\epsilon^{-2bn} & \text{on } [\epsilon, \|z\|_K]. \end{cases}
\]
and let \( p_\varepsilon(u) \) be a polynomial which uniformly approximates \( h_\varepsilon(u) \) within \( \varepsilon \) on the interval \([0, ||z||_K]\). Consider the function \( q_\varepsilon(z) = (z^m)^a (z^n)^{-b} p_\varepsilon(||z||) \). Clearly \( q_\varepsilon \in B \). If \( z_0 \in K \) and \( ||z_0|| < \varepsilon \), then

\[
|q_\varepsilon(z_0) - z_0| \leq |q_\varepsilon(z_0)| + |z_0| < \varepsilon^{am-bn}(h_\varepsilon(||z_0||) + \varepsilon) + \varepsilon = 2\varepsilon + \varepsilon^{am-bn+1}.
\]

If \( z_0 \in K \) and \( ||z_0|| \geq \varepsilon \), then

\[
|q_\varepsilon(z_0) - z_0| = |(z_0^m)^a (z_0^n)^{-b} \cdot |p_\varepsilon(||z_0||)| - h_\varepsilon(||z_0||)| < \varepsilon||z||^{am-bn}.
\]

Hence

\[
||q_\varepsilon - z||_K \to 0
\]
as \( \varepsilon \to 0 \), and so \( z \in B \). In completely analogous fashion, if we let

\[
\tilde{p}_\varepsilon(u) = \begin{cases}
1/\varepsilon^{2n-2} & \text{on } [0, \varepsilon], \\
1/u^{2n-2} & \text{on } [\varepsilon, ||z||_K],
\end{cases}
\]

let \( \tilde{p}_\varepsilon(u) \) be a polynomial which uniformly approximates \( \tilde{h}_\varepsilon(u) \) within \( \varepsilon \) on the interval \([0, ||z||_K]\), and put \( \tilde{q}_\varepsilon(z) = z^n z^{-n-1} \tilde{p}_\varepsilon(||z||) \), then \( \tilde{q}_\varepsilon \in B \) and \( ||\tilde{q}_\varepsilon - z||_K \to 0 \) as \( \varepsilon \to 0 \), so \( z \in B \). This establishes our claim that \( B = C(K) \). Since any \( g \in C(K) \) is thus uniformly approximable on \( K \) by polynomials in \( z^m \) and \( z^n \), it follows that \( g \circ f \) is uniformly approximable on \( X \) by polynomials in \( f^m \) and \( f^n \). Hence \( A \supseteq \{ g \circ f : g \in C(f(X)) \} \). The remaining assertions are obvious.

**Corollary.** Let \( K \) be a compact subset of the complex plane. Then \( R(K) = C(K) \) iff \( z^n \in R(K) \) for some \( n \in \mathbb{N} \).

**Proof.** Immediate from the theorem.

**Remark.** An analysis of the proof of Theorem 1 shows that the closed subalgebra of \( C(X) \) generated by \( f^m \) and \( f^n \) contains both \( f \) and \( f^n \); that is, \( \{ f^m, f^n \} \) and \( \{ f, f^n \} \) generate the same closed subalgebra of \( C(X) \). Consequently, the following extended Stone-Weierstrass theorem holds:

Let \( X \) be a compact Hausdorff space and let \( A \) be a subalgebra of \( C(X) \) such that:

1. if \( x_0 \in X \), then there exists \( f \in A \) with \( f(x_0) \neq 0 \),
2. \( A \) separates the points of \( X \),
3. if \( f \in A \), then \( f^n \in A \) for some \( n \in \mathbb{N} \) depending on \( f \).

Then \( A \) is dense in \( C(X) \).

**Theorem 2.** Let \( X \) be a compact Hausdorff space, and let \( f \in C(X) \). Let \( \alpha > 0 \). If \( A \) is the subalgebra of \( C(X) \) generated by \( f, |f|^\alpha \), and the constant functions, then \( \text{Re} A \) (i.e., the real parts of functions in \( A \)) is dense in
\{g \circ f: g \in C_\mathbb{R}(f(X))\}. In particular, if f separates the points of X, then A is a dirichlet algebra on X (i.e., Re A is dense in C_\mathbb{R}(X)).

**Proof.** Let \( K = f(X) \), and let \( B \) be the closed subalgebra of \( C(K) \) generated by \( z, |z|^{\alpha} \), and the constant functions. We claim Re \( B \) is dense in \( C_\mathbb{R}(K) \). As in the proof of Theorem 1, \( |z|^{\alpha} \in B \) implies \( |z| \in B \). If \( j \) and \( k \) are nonnegative integers with \( j \geq k \), then \( \text{Re} (z^j z^k) = \text{Re} (z^{j-k} |z|^{2k}) \in \text{Re} B \), and if \( j < k \), then

\[
\text{Re} (z^j z^k) = \text{Re} (\overline{z^j z^k}) = \text{Re} (z^{-k} |z|^{2j}) \in \text{Re} B.
\]

Analogously, \( \text{Im} (z^j z^k) = \text{Re} (-iz^j z^k) \) implies that \( \text{Im} (z^j z^k) \in \text{Re} B \). By the Stone-Weierstrass theorem, the algebra generated by \( z, \overline{z} \), and the constants is dense in \( C(K) \). The density of \( \text{Re} B \) in \( C_\mathbb{R}(K) \) follows immediately, establishing our claim. Since any \( g \in C_\mathbb{R}(K) \) is thus uniformly approximable on \( K \) by the real parts of polynomials in \( z \) and \( |z|^{\alpha} \), \( g \circ f \) is uniformly approximable on \( X \) by the real parts of polynomials in \( f \) and \( |f|^{\alpha} \). This completes the proof.

**Corollary.** Let \( K \) be a compact subset of the complex plane. Then \( R(K) = C(K) \) iff \( |z|^{\alpha} \in R(K) \) for some \( \alpha > 0 \).

**Proof.** By the preceding theorem, \( |z|^{\alpha} \in R(K) \) implies that \( R(K) \) is a dirichlet algebra on \( K \). This in turn implies that every point of \( K \) is a peak point for \( R(K) \). By a theorem of Bishop [1, Theorem 3.3.3], we conclude \( R(K) = C(K) \).

**References**