AN INEQUALITY FOR POSITIVE DEFINITE VOLterra KERNELS

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Abstract. We deduce an inequality satisfied by certain positive definite Volterra kernels. This inequality yields a new theorem on the asymptotic behavior of the bounded solutions of a Volterra equation.

1. Introduction. Fourier transform methods have recently been used to get very sharp results on the asymptotic behavior of the bounded solutions of the nonlinear Volterra equation

\[ x'(t) + \int_{[0,t]} g(x(t - s)) \, d\mu(s) = f(t) \quad (t \in \mathbb{R}^+); \quad x(0) = x_0; \]

see e.g. [1], [3], [5], [6] and [7]–[10] (we write \( \mathbb{R}^+ \) for the interval \([0, \infty)\)). This approach is possible whenever the function \( f \) is integrable and the kernel \( \mu \) is a positive definite measure, i.e. the distribution Fourier transform \( \hat{\mu} \) of \( \mu \) has a nonnegative real part (no integrability of \( f \) is required in [3]). The treatment in [1], [5] and [6] is based on the notion of a “strongly positive definite” kernel: There exists \( \epsilon > 0 \) such that \( \text{Re} \, \hat{\mu}(\omega) \geq \epsilon (1 + \omega^2)^{-1} \) (\( \omega \in \mathbb{R} \)) (interpret the inequality in the distribution sense if \( \hat{\mu} \) does not have a classical Fourier transform). This requirement has been relaxed in [7] and [8] to “strict positive definiteness”: \( \text{Re} \, \hat{\mu} \) is strictly positive everywhere in the appropriate sense. The question how zeros of \( \text{Re} \, \hat{\mu} \) affect the asymptotic behavior of the solutions of (1) is studied in [9] and [10].

The purpose of this paper is to present yet another Fourier transform condition, which overlaps the notion of strict positive definiteness. We consider kernels \( \mu \) with a finite total variation satisfying

\[ \text{There exists } \alpha > 0 \text{ such that} \]

\[ \alpha \, \text{Re} \, \hat{\mu}(\omega) \geq |\hat{\mu}(\omega)|^2 \quad (\omega \in \mathbb{R}). \]

This assumption is clearly stronger than positive definiteness, but it does not imply strict positive definiteness since \( \text{Re} \, \hat{\mu}(\omega) = 0 \) is possible in some cases (a specific example is given following Theorem 2 below). On the other hand, not all strictly positive definite kernels satisfy (2).

The usefulness of (2) is due to the fact that it yields

Lemma 1. Let \( \mu \) be a finite (real) Borel measure on \( \mathbb{R}^+ \), and suppose that (2)

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holds, where \( \hat{\mu}(\omega) = \int_{R^+} e^{-i\omega t} d\mu(t) \) (\( \omega \in R \)). Then

\[
\int_{[0,T]} \left| \int_{[0,T]} q(t-s) d\mu(s) \right|^2 dt 
\leq \alpha \int_{[0,T]} q(t) \int_{[0,T]} q(t-s) d\mu(s) dt 
\]

for every \( T \in R^+ \), and for every (real) \( \varphi \in C[0,T] \).

Inequality (3) is the key ingredient in the proof of

**Theorem 1.** (i) Let \( g \in C(R) \), \( f \in L^1(R^+) \), and let \( \mu \) satisfy the assumption of Lemma 1. Moreover, suppose that \( x \) is a bounded solution of (1) on \( R^+ \). Then \( x^\prime - f \in L^2(R^+) \).

(ii) Let the assumption of (i) hold. In addition, suppose that \( \int_{R^+} |\mu([0,s])| ds < \infty \). Then

\[
\lim_{t \to \infty} \left\{ x(t) + g(x(t)) \int_{R^+} \mu([0,s]) ds \right\} = x_0 + \int_{R^+} f(s) ds. 
\]

(iii) Let the assumption of (i) hold. In addition, suppose that

\[
\sup_{T>0} \int_{[0,T]} \mu([0,s]) ds = \infty. 
\]

Then \( g(x(t)) \to 0 \) (\( t \to \infty \)).

Theorem 1 motivates us to study the following question: Which classical kernels satisfy (2)?

**Theorem 2.** Condition (2) is satisfied in each of the following cases:

(i) \( b(t) = \mu([0,t]) \) is nonnegative and nonincreasing on \( R^+ \).

(ii) \( d\mu(t) = a(t) dt \) (\( t \in R^+ \)), where \( a \in L^1(R^+) \cap BV(R^+) \) is strongly positive definite, i.e. there exists \( \epsilon > 0 \) such that \( \int_{R^+} \cos(\omega t)a(t) dt \geq \epsilon(1 + \omega^2)^{-1} \) (\( \omega \in R \)).

(iii) \( d\mu(t) = a(t) dt \) (\( t \in R^+ \)), where \( a \) and \( -a' \) are nonnegative and convex on \( (0, \infty) \), and \( a \in L^1(R^+) \).

Using Theorem 2(i) we can give a nontrivial example of a kernel which satisfies (2), but which is not strictly positive definite. Take \( b(t) = 1 \) (\( 0 \leq t \leq 1 \)), \( b(t) = 0 \) (\( t > 1 \)), and define \( \mu \) as in (i). Then (2) holds. However, calculating \( \text{Re} \hat{\mu}(\omega) = 1 - \cos(\omega) = 0 \) (\( \omega = 0, \pm 2\pi, \ldots \)), we find that \( \text{Re} \hat{\mu} \) is not strictly positive everywhere, i.e. \( \mu \) is not strictly positive definite.

The result one gets by combining Theorem 1 and 2(i) is contained in a paper by Londen [4], which together with [7] has been the main source of inspiration for this study. In [4] Londen deduces an inequality of type (3) for the kernel in Theorem 2(i), and employs it to greatly simplify his original proof of [2, Theorem 1]. Recently Staffans [10] has given yet another proof of (a slightly weakened version of) [2, Theorem 1]. The argument in [10] is, however, more complicated than the one in [4] and than the one presented here.

Applying Theorems 1 and 2(ii) one can add the conclusion \( x^\prime - f \in L^2(R^+) \)
to [6, Theorem 1(ii)] and get an alternative proof of [6, line (1.6)] in the special case when \( a \in L^1(R^+) \cap BV(R^+) \).

Combining Theorems 1 and 2(iii) we get

**Corollary 1.** Let \( g \in C(R) \), \( f \in L^1(R^+) \), \( a \in L^1(R^+) \), and suppose that \( a, -a' \) are nonnegative and convex. Then every bounded solution \( x \) of

\[
x'(t) + \int_{[0,t]} g(x(t-s))a(s)ds = f(t) \quad (t \in R^+)
\]

satisfies \( x' - f \in L^2(R^+) \). If moreover \( a \not\equiv 0 \), then \( g(x(t)) \to 0 \) \((t \to \infty)\).

The second conclusion of Corollary 1 is well known, and is valid under weaker assumptions on \( a \) than those given here (see e.g. [6, Theorem 1(ii) and Corollary 2.2]). However, the first conclusion is new.

2. **Proof of Lemma 1.** The argument presented below is quite similar to the one in [7, §4], and therefore we omit a detailed motivation of each step (the extension to \( R \) of the function \( a \) in [7, Lemma 2] is replaced by the extension used in [8, Lemma 1.1]).

Define

\[
\varphi_T = \chi_{[0,T]} \varphi; \quad \hat{\varphi}_T(\omega) = \int_{[0,T]} e^{-i\omega t} \varphi(t) dt.
\]

Then one has

\[
\alpha \int_{[0,T]} \varphi(t) \int_{[0,t]} \varphi(t-s) d\mu(s) dt = \frac{\alpha}{2\pi} \int_R |\hat{\varphi}_T(\omega)|^2 \Re \hat{\mu}(\omega) d\omega
\]

\[
\geq \frac{1}{2\pi} \int_R |\hat{\varphi}_T(\omega)\hat{\mu}(\omega)|^2 d\omega = \int_R \left| \int_{[t-T,t]} \varphi(t-s) d\mu(s) \right|^2 dt
\]

\[
\geq \int_{[0,T]} \left| \int_{[0,t]} \varphi(t-s) d\mu(s) \right|^2 dt,
\]

where the first inequality follows from (2), and the other steps use only elementary properties of Fourier transforms.

3. **Proof of Theorem 1.** It is well known (argue as in [7, §2]) that the assumption of Theorem 1 yields

\[
\sup_{T>0} \int_{[0,T]} g(x(t)) \int_{[0,t]} g(x(t-s)) d\mu(s) dt < \infty.
\]

Hence by Lemma 1,

\[
\int_{R^+} \left| \int_{[0,t]} g(x(t-s)) d\mu(s) \right|^2 dt < \infty,
\]

which together with (1) gives \( x' - f \in L^2(R^+) \) and verifies (i).

To prove part (ii) one applies [10, Theorem 4.2] (note that it follows from part (i) that every \( y \in F(x) \) is a constant, which is required in [10, Theorem 4.2]). Part (iii) is demonstrated using a minor modification of the arguments in
the second paragraph of the proof of [10, Theorem 4.1].

4. Proof of Theorem 2(i). By Hölder’s inequality,
\[
|\text{Im } \hat{\mu}(\omega)|^2 = \left| \int_{(0,\infty)} \sin(\omega t) \, d\mu(t) \right|^2 \\
\leq -[b(0) - b(\infty)] \int_{(0,\infty)} \sin^2(\omega t) \, d\mu(t).
\]

Hence
\[
2[b(0) - b(\infty)] \text{Re } \hat{\mu}(\omega) - |\text{Im } \hat{\mu}(\omega)|^2 
\geq 2[b(0) - b(\infty)] \left\{ b(0) + \int_{(0,\infty)} [\cos(\omega t) + \frac{1}{2} \sin^2(\omega t)] \, d\mu(t) \right\} \\
= 2[b(0) - b(\infty)] \left\{ b(0) + \int_{(0,\infty)} [1 - \frac{1}{2}(1 - \cos(\omega t))^2] \, d\mu(t) \right\} \\
\geq 2[b(0) - b(\infty)] \left\{ b(0) + \int_{(0,\infty)} \, d\mu(t) \right\} \\
= 2[b(0) - b(\infty)] b(\infty) > 0.
\]
Combining this with the trivial estimate \( \text{Re } \hat{\mu}(\omega) \leq 2b(0) - b(\infty) \) one obtains (2) with \( \alpha = 4b(0) - 3b(\infty) \).

5. Proof of Theorem 2(ii). The fact that \( a \in BV(R^+) \) gives
\[
|\hat{\mu}(\omega)| = \left| \int_{R^+} e^{-i\omega t} a(t) \, dt \right| = \frac{1}{\omega} \left| \int_{R^+} (1 - e^{-i\omega t}) \, da(t) \right| \\
\leq 2A/\omega \quad (\omega \in R, \omega \neq 0),
\]
where \( A \) is the total variation of \( a \) on \( R^+ \). This together with \( |\hat{\mu}(\omega)| \leq \int_{R^+} |a(t)| \, dt \) yields the existence of a constant \( \gamma \) such that
\[
|\hat{\mu}(\omega)|^2 \leq \gamma(1 + \omega^2)^{-1} \quad (\omega \in R).
\]
The strong positive definiteness of \( a \) then implies (2) with \( \alpha = \gamma/\varepsilon \).

6. Proof of Theorem 2(iii). We first notice that one can use the monotonicity of \( a \) together with \( a \in L^1(R^+) \) to get
\[
ta(t) \leq 2 \int_{[t/2,t]} a(s) \, ds = O(1) \quad (t \to 0^+, t \to \infty),
\]
and then one can show inductively
\[
|t^{k+1}a^{(k)}(t)| \leq 2t^k \int_{[t/2,t]} |a^{(k)}(s)| \, ds \\
\leq 2t^k |a^{(k-1)}(t/2)| = O(1) \quad (t \to 0^+, t \to \infty)
\]
for \( k = 1, 2 \). These estimates justify the integrations by parts that are performed below.

We integrate by parts twice to get
\[ \text{Im} \hat{\mu}(\omega) = \frac{1}{\omega} \int_{\mathbb{R}^+} t^{-2} \left( t - \frac{1}{\omega} \sin(\omega t) \right) t^2 a''(t) \, dt \quad (\omega \neq 0). \]

Hence by Hölder's inequality and the fact that
\[ \int_{\mathbb{R}^+} t^2 a''(t) \, dt = 2 \int_{\mathbb{R}^+} a(t) \, dt = \frac{\text{def}}{2} 2A, \]

one has
\[ |\text{Im} A|^2 \leq 2A \omega^{-2} \int_{\mathbb{R}^+} t^{-2} \left( t - \frac{1}{\omega} \sin(\omega t) \right)^2 a''(t) \, dt \quad (\omega \neq 0). \]

One more integration by parts together with a change of variable yields
\[ |\text{Im} A|^2 \leq \frac{2}{\omega^3} \int_{0}^{\infty} h(\omega t) a''(t) \, d\omega \quad (\omega \neq 0), \]

where \( h(t) = \int_{[0,1]} \left( 1 - s^{-1} \sin(s) \right)^2 \, ds. \)

On the other hand, one can integrate \( \text{Re} \, \hat{\mu} \) by parts three times to get
\[ \text{Re} \, \hat{\mu}(\omega) = -\frac{\omega^{-3}}{3} \int_{0}^{\infty} (\omega t - \sin(\omega t)) a''(t) \, d\omega \quad (\omega \neq 0). \]

We claim that
\[ h(t) \leq 2(t - \sin(t)) \quad (t \in \mathbb{R}^+). \]

Assume this for the moment. Then clearly (4) and (5) together with the convexity of \(-a'\) imply \( |\text{Im} \hat{\mu}(\omega)|^2 \leq 4A \, \text{Re} \, \hat{\mu}(\omega) \) \((\omega \neq 0)\). The same inequality is trivially true for \( \omega = 0 \). Combining this with the fact that \( |\text{Re} \, \hat{\mu}(\omega)| \leq A \) one gets (2) with \( \alpha = 5A \). Thus it only remains to prove (5).

Using the power series expansion of \( \sin(t) \) one can easily check that
\[ h(t) \leq t^2/180, t - \sin(t) \geq t^3/6 - t^5/120 (t \in \mathbb{R}^+). \]

Hence, in particular, \( h(t) \leq (2(t - \sin(t)) (t < 2) \). For the remaining values of \( t \), i.e. for \( t > 2 \), we estimate
\[ h(t) \leq h(2) + (t - 2) \sup_{s \in \mathbb{R}^+} (1 - s^{-1} \sin(s))^2 \]
\[ < 32/180 + 2(t - 2) < 2(t - 1) \leq 2(t - \sin(t)). \]

This yields (6), and completes the proof of Theorem 2(iii).

7. A final comment. We have throughout assumed that the kernel \( \mu \) has a finite total variation (or that the function \( a \) in Theorem 2(ii)–(iii) is integrable). This condition can be weakened somewhat, but there is a built-in restriction in (2) which limits the class of kernels that can be treated with the method presented above. Note that (2) implies \( |\hat{\mu}(\omega)| \leq \alpha \) \((\omega \in \mathbb{R})\), i.e. the Fourier transform of the kernel must be a bounded function. This necessary condition is in fact also sufficient, i.e. Lemma 1 and Theorem 1 are true for any positive definite measure \( \mu \) whose Fourier transform is a bounded function satisfying (2).

Notes added in proof. 1. One version of Lemma 1 is contained in the


**References**


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