CYCLICALLY MONOTONE LINEAR OPERATORS

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ABSTRACT. A linear operator on a complex Hilbert space $\mathcal{H}$ is called $n$-cyclically monotone if for each sequence $x_0, x_1, \ldots, x_{n-1}, x_n = x_0$ of $n$ elements in $\mathcal{H}$, $\sum_{j=0}^{n-1} \text{Re}(T_{x_j} x_{j+1}) > 0$. We show that $T$ is $n$-cyclically monotone if and only if $|\text{Arg}(Tx, x)| < \pi/n$, $\forall x \in \mathcal{H}$. If $T_m$ and $T_n$ are $m$- and $n$-cyclically monotone operators, then the spectrum of the product $T_m T_n$ lies in the sector $\{z \in \mathbb{C}: |\text{Arg} z| < \pi/m + \pi/n\}$.

1. Introduction. Let $H$ denote a real Hilbert space with inner product $(\cdot,\cdot)$. The following is a simplified version of [1, Theorem 3]: Let $f$ and $f_1$ be two continuous (not necessarily linear) functions on $H$, mapping bounded subsets into bounded subsets, such that (i) $f$ is monotone, i.e., $(f(x) - f(y), x - y) > 0$, $\forall x, y \in H$, (ii) $f_1$ is tricyclically monotone, i.e., $(f_1(x), x - y) + (f_1(y), y - z) + (f_1(z), z - x) > 0$, $\forall x, y, z \in H$. Then $I + ff_1$ is a homeomorphism.

This paper is motivated by the theorem above and we shall restrict our discussion to the elements in $B(\mathcal{H})$, the set of bounded linear operators on a complex Hilbert space $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is called $n$-cyclically monotone, $n$ an integer greater than one, if for each sequence $x_0, x_1, x_2, \ldots, x_{n-1}, x_n = x_0$ of $n$ points in $\mathcal{H}$, $\sum_{j=0}^{n-1} \text{Re}(T_{x_j} x_{j+1}) > 0$. A 2-cyclically monotone operator will be called accretive [5, p. 279]. The concept of the cyclically monotone operators was first introduced by R. T. Rockafellar [6]. According to [6], an $n$-cyclically monotone operator should be called monotone of degree $(n - 1)$; however, we justify our definition with the following theorem: $T$ is $n$-cyclically monotone if and only if

$$|\text{Arg}(Tx, x)| < \pi/n, \quad \forall x \in \mathcal{H}.$$ 

In the last section of this paper we show that if $T$ is accretive and $T_1$ is 3-cyclically monotone, then for each $\lambda$ in the spectrum of $TT_1$, $|\lambda| \leq \pi/2 + \pi/3$; consequently $I + TT_1$ is invertible.

2. Notation and preliminaries. Let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{R}^+$ denote the set of complex, real and nonnegative real numbers, respectively. Let $\Omega$, $\Omega_1 \subset \mathbb{C}$, $\Omega \cdot \Omega_1 = \{zz_1: z \in \Omega, z_1 \in \Omega_1\}$; $\text{Cl}(\Omega)$ denotes the closure and $\text{Co}(\Omega)$ the convex hull of $\Omega$. For $\alpha, \beta \in \mathbb{R}$, $0 \leq \beta - \alpha < 2\pi$, $\Sigma(\alpha, \beta)$ denotes the closed sector \{z \in \mathbb{C}: \alpha \leq \arg z \leq \beta\}. For $\alpha \in \mathbb{R}$, $0 \leq \alpha < \pi$, $\Sigma(\alpha)$ denotes the symmetric sector $\Sigma(-\alpha, \alpha)$.

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For $T \in \mathfrak{B}(\mathcal{K})$, $\Re T = (T + T^*)/2$ and $\Im T = (T - T^*)/2i$; $\sigma(T)$ denotes the spectrum and $W(T)$ the numerical range of $T$, $W(T) = \{(Tx, x) : \|x\| = 1\}$. $T$ is called nonnegative if $W(T) \subseteq [0, \infty)$. We define $A(T) = \text{Cl}(\{(Tx, x)\})$. Since $A(T) = \mathbb{R}^+ \cdot \text{Cl}(W(T))$ and the numerical range of an operator is convex, either $A(T) = \mathbb{C}$ or $A(T) = \Sigma(\alpha, \beta)$ with $\beta - \alpha \leq \pi$.

If $\mathcal{K}$ is finite dimensional and $0 \in W(T)$, then $A(T)$ coincides with the angular field introduced in [10].

**Lemma 1.** Let $T, S \in \mathfrak{B}(\mathcal{K})$. If $S$ is invertible, then $A(T) = A(S^*TS)$.

**Lemma 2** [5, VI-§1.2]. Let $T \in \mathfrak{B}(\mathcal{K})$ and $\alpha \in [0, \pi/2)$; then the following three statements are equivalent:

1. $A(T) \subseteq \Sigma(\alpha)$;
2. $|\langle \Im Tx, x \rangle| \leq \tan(\alpha)|\langle \Re Tx, x \rangle|$, $\forall x \in \mathcal{K}$;
3. $|\langle \Im Ty, y \rangle| \leq \tan(\alpha)|\langle \Re Tx, x \rangle(\Re Ty, y)\rangle|^{1/2}$, $\forall x, y \in \mathcal{K}$.

Furthermore, each of these conditions implies

4. $\|Tx\|^2 \leq (1 + \tan(\alpha))^2 \|\Re T\|(\Re Tx, x)$, $\forall x \in \mathcal{K}$.

Let $S(n)$ denote the $n$-by-$n$ backward-shift matrix, i.e., $S(n) = (\delta_{i+1,j})_{n \times n}$. Let $R(n) = (I - S(n))^{-1}$, then $R(n)$ is the $n$-by-$n$ matrix with 1's on and above the diagonal and 0's below the diagonal.

**Lemma 3.** $A(R(n)) = \Sigma(\pi/2 - \pi/(n + 1))$.

**Proof.** Since $R(n)$ is a real matrix, $A(R(n)) = A(I - S(n))$. The result follows if we show that $W(S(n))$ is a disc centered at 0 with radius $\cos(\pi/n + 1)$. It is easy to see that $W(S(n))$ is a disc centered at 0. The numerical radius of $S(n)$ is the spectral radius of $\Re S(n)$. Put $U_m(\lambda) = \det(2\lambda - S(m) - S(m)^*)$, $m = 2, 3, \ldots$. If we define $U_0(\lambda) = 1$ and $U_1(\lambda) = 2\lambda$, then $U_m(\lambda) = 2\lambda U_{m-1}(\lambda) - U_{m-2}(\lambda)$, $m = 2, 3, \ldots$. We notice that $U_m(\lambda)$ satisfies the recurrence relations and initial conditions of the Chebyshev polynomial of the second kind [9, p. 128]. Thus

$$U_m(\lambda) = \sin((m + 1)\arccos(\lambda))/\sin(\arccos(\lambda)).$$

Consequently the numerical radius of $S(n)$ is $\cos(\pi/n + 1)$.

**Proposition [2].** Let $S, T \in \mathfrak{B}(\mathcal{K})$ and let $S \otimes T$ denote the tensor product acting on the product space $\mathcal{K} \otimes \mathcal{K}$. Then $\sigma(S \otimes T) = \sigma(S) \cdot \sigma(T)$.

**Corollary.** Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces. For $T_j \in \mathfrak{B}(\mathcal{K}_j)$, $j = 1, 2$, $\sigma(T_1 \otimes T_2) = \sigma(T_1) \cdot \sigma(T_2)$.

**Lemma 4** [8]. Let $T_j \in \mathfrak{B}(\mathcal{K}_j)$ be a normal operator, $j = 1, 2$. Then $\text{Cl}(W(T_1 \otimes T_2)) = \text{Cl}(\text{Co}(W(T_1)) \cdot W(T_2)))$.

**Proof.** $T_1 \otimes T_2$ is also normal.

L.H.S. $= \text{Co}(\sigma(T_1 \otimes T_2))$ = $\text{Co}(\sigma(T_1) \cdot \sigma(T_2))$ by Corollary

$= \text{Co}(\text{Co}(\sigma(T_1)) \cdot \text{Co}(\sigma(T_2)))$

$= \text{Co}(\text{Cl}(W(T_1)) \cdot \text{Cl}(W(T_2))) = \text{R.H.S.}$.
3. Characterizations of cyclically monotone linear operators.

**Theorem 1.** Let $T \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent.

1. $T$ is $n$-cyclically monotone.
2. For every sequence $y_1, \ldots, y_{n-1}$ of $(n-1)$ points in $\mathcal{H}$,
   \[
   \sum_{j=1}^{n-1} \text{Re} \left( Ty_j, \sum_{k=1}^{j} y_k \right) > 0.
   \]
3. The operator $R(n-1) \otimes T$ on $\mathbb{C}^{n-1} \otimes \mathcal{H}$ is accretive.
4. $A(T) \subset \Sigma(\pi/n)$.

**Proof.** (1) $\iff$ (2).

\[
\sum_{j=0}^{n-1} (Tx_j, x_j - x_{j+1}) = \sum_{j=1}^{n-1} (Tx_j - Tx_{j-1}, x_j - x_0)
= \sum_{j=1}^{n-1} \left( Ty_j, \sum_{k=1}^{j} y_k \right),
\]
where $y_k = x_k - x_{k-1}$.

(2) $\iff$ (3).

\[
\sum_{j=1}^{n-1} \left( Ty_j, \sum_{k=1}^{j} y_k \right) = \sum_{j=1}^{n-1} \left( y_j, \sum_{k=1}^{j} T^* y_k \right)
= \left[ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{array} \right] \left[ \begin{array}{c} T^* y_1 \\ T^* y_1 + T^* y_2 \\ \vdots \\ T^* y_1 + T^* y_2 + \cdots + T^* y_{n-1} \end{array} \right]
= \left[ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{array} \right] \left[ \begin{array}{cc} T^* & \bigodot \\ T^* T^* & \ldots \end{array} \right] \left[ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{array} \right]
= R(n-1) \otimes T \left[ \begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{array} \right].
\]

(3) $\iff$ (4) is an immediate consequence of Lemma 3 and

**Theorem 2.** Let $T_j \in \mathcal{B}(\mathcal{H}_j)$ and $A(T_j) = \Sigma(\alpha_j, \beta_j), j = 1, 2$. Suppose either

(i) $A(T_1 \otimes T_2) \neq \mathcal{H}$, or

(ii) $(\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < \pi$; then $A(T_1 \otimes T_2) = \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2)$.
Proof. Since it is always true that $W(T_1 \otimes T_2) \supset W(T_1) \cdot W(T_2)$,

$$A(T_1 \otimes T_2) \supset \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2).$$

Consequently, assumption (i) implies assumption (ii).

To show that $A(T_1 \otimes T_2) \subset 2(a_1 + a_2, b_1 + b_2)$, we need only to establish the case $a_j = -b_j$, i.e., $A(T_j) = \Sigma(b_j)$, $j = 1, 2$. Write $T_j = \text{Re } T_j + \text{Im } T_j$, $j = 1, 2$. Since $\text{Re } T_j$ is nonnegative, it has a nonnegative square root $Q_j$ [5, Theorem V.3.35(iv)]. Furthermore, if we assume that $\text{Re } T_j$ is invertible, then $T_j = Q_j N_j Q_j^{-1}$, where $N_j$ is the normal operator $I + iQ_j^{-1} (\text{Im } T_j) Q_j^{-1}$, $j = 1, 2$. Thus $T_1 \otimes T_2 = (Q_1 \otimes Q_2)(N_1 \otimes N_2)(Q_1 \otimes Q_2)$.

$$A(T_1 \otimes T_2) = A(N_1 \otimes N_2) \text{ by Lemma 1}$$

$$= \mathbb{R}^+ \cdot \text{Cl}(W(N_1 \otimes N_2))$$

$$= \mathbb{R}^+ \cdot \text{Cl}(\text{Co}(W(N_1)W(N_2))) \text{ by Lemma 4}$$

$$= \Sigma(\beta_1 + \beta_2) \text{ by assumption (ii).}$$

Thus the theorem is proved if both $\text{Re } T_1$ and $\text{Re } T_2$ are invertible. In general, we have $A((T_1 + \epsilon) \otimes (T_2 + \epsilon)) \subset \Sigma(\beta_1 + \beta_2)$ for each $\epsilon > 0$. $\text{Cl}(W(\cdot))$ is continuous with respect to the uniform operator topology [3, Problem 175]; we let $\epsilon$ tend to 0 and obtain $A(T_1 \otimes T_2) \subset \Sigma(\beta_1 + \beta_2)$. □

Theorem 1 answers the conjecture raised in [6, p. 500]. The following corollary is a complex linear operator version of [6, Theorem 1] and [7, Theorem 24.8].

Corollary 1. For $T \in \mathfrak{S} (\mathcal{H})$, $T$ is nonnegative if and only if $T$ is $n$-cyclically monotone, $n = 2, 3, \ldots$.

Remarks. Since the concept of an $n$-cyclically monotone operator is in essence a finite dimensional one, Theorem 1 can be rephrased for the cases of unbounded operators or sectorial sesquilinear forms [5, §VI-1.2]. An $n$-cyclically monotone linear operator, if defined on the whole Hilbert space, is necessarily bounded [5, Theorem V.3.4].

4. Spectra of products. In this section we study the spectrum location of the product of two operators.

Theorem 3 [10, Theorem 2], [11, Theorem 1]. Let $S, T \in \mathfrak{S} (\mathcal{H})$. If $0 \not\in \text{Cl}(W(T))$, then $\{\sigma(ST) \cup \sigma(TS)\} \subset \text{Cl}(W(S))/\text{Cl}(W(T^{-1}))$.

Proof. We note that the nonzero elements of $\sigma(ST)$ and $\sigma(TS)$ are the same [3, Problem 61], and $0 \in \text{Cl}(W(T))$ if and only if $0 \in \text{Cl}(W(T^{-1}))$. If $0 \in \sigma(ST - \lambda)$, then

$$0 \in \sigma(S - \lambda T^{-1}) \subset \text{Cl}(W(S - \lambda T^{-1}))$$

$$\subset \text{Cl}(W(S)) - \lambda \cdot \text{Cl}(W(T^{-1})). \quad \Box$$

Thus for an $m$-cyclically monotone operator $S$ and an $n$-cyclically monotone operator $T$, $\{\sigma(ST) \cup \sigma(TS)\} \subset \Sigma(\pi/m + \pi/n)$ if $0 \not\in \text{Cl}(W(S))$ or

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0 \not\in \text{Cl}(W(T)). \text{ We conclude this paper by showing that the last assumption is not necessary.}

**Theorem 4** (cf. [4]). \textit{Let} \( S, T \in \mathcal{B} (\mathcal{H}) \) \textit{with} \( S \) \textit{accretive and} \( T \) \textit{satisfying the condition:}

(\*) \textit{There exists a constant} \( d > 0 \) \textit{such that} \( \text{Re}(Tx, x) > d \| Tx \|^2, \forall x \in \mathcal{H} \).

\textit{Then} \( (-\infty, 0) \cap \{ \sigma(ST) \cup \sigma(TS) \} = \emptyset \).

**Proof.** Let \( \lambda \) be a point in the approximate point spectrum of \( ST \), i.e., there exists a sequence \( \{x_n\} \) of unit vectors such that \( \|(\lambda - ST)x_n\| \to 0 \). Since \( (\lambda x_n, Tx_n) - (STx_n, x_n) \to 0 \) and \( S \) is accretive, \( \lim \inf \text{Re}(\lambda x_n, Tx_n) = \lim \inf \text{Re}(STx_n, x_n) \geq 0 \). If we assume \( \lambda < 0 \), then \( \lim \sup \text{Re}(x_n, Tx_n) < 0 \). By (\*), \( \text{Re}(x_n, Tx_n) > d \| Tx_n \|^2 \); consequently, \( \| Tx \| \to 0 \) and this contradicts \( \lambda \neq 0 \). Thus the approximate point spectrum of \( ST \) has no negative numbers, and therefore the boundary of \( \sigma(ST) \) has no negative numbers [3, Problem 63]. Hence \( (-\infty, 0) \cap \sigma(ST) = \emptyset \). \( \square \)

For \( T \in \mathcal{B} (\mathcal{H}), \) if \( A(T) \subset \Sigma(\alpha) \text{ with } \alpha < \pi/2, \) then \( T \) satisfies (\*) by the last part of Lemma 2. However, the converse does not hold; the example in [4, p. 309] is also valid for the complex case.

**Theorem 5.** Let \( T_j \in \mathcal{B} (\mathcal{H}) \) \textit{with} \( A(T_j) = \Sigma(\alpha_j, \beta_j), j = 1, 2 \). \textit{Suppose} \( (\beta_1 - \alpha_1) + (\beta_2 - \alpha_2) < 2\pi; \) \textit{then}

\[ \{ \sigma(T_1 T_2) \cup \sigma(T_2 T_1) \} \subset \Sigma(\alpha_1 + \alpha_2, \beta_1 + \beta_2). \]

**Proof.** Consider the operators \( e^{i\theta}T_j, j = 1, 2; \) vary the real numbers \( \theta_1 \) and \( \theta_2 \) and apply Theorem 4. \( \square \)

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