LIMITS OF SUCCESSIVE CONVOLUTIONS

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ABSTRACT. On an arbitrary compact, zero-dimensional, Abelian group, if μ₀, μ₁, ... is a sequence of probability measures, a condition on these measures is given which is necessary and sufficient for each of the sequences μ₁, μ₁ * μ₂, ... of successive convolutions to converge to Haar measure in the weak-star topology. Some simple consequences of the theorem are noted.

Let G be a compact, Abelian group and μ a Borel probability measure on G. If we denote by μⁿ the nth convolution power of μ, the following result is well known.

**Theorem [1, Theorem 8].** μⁿ converges to the normalized Haar measure λ₆ in the weak-star topology if and only if the support of μ is not contained in a coset of any proper closed subgroup of G.

In this paper we shall be concerned primarily with zero-dimensional, compact, Abelian groups. For such a group G we consider, in place of μⁿ, successive convolutions of the form μᵢ * μᵢ₊₁ * ... * μᵢ₊ₘ (henceforth abbreviated μᵢₘ), where μ₁, μ₂, ... is any sequence of Borel probability measures on G; we derive a necessary and sufficient condition for μᵢₘ to converge to λ₆ in the weak-star topology for every k.

For background material, including a more complete discussion of convolutions, the reader is referred to [3].

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If G is any compact, Abelian group, C(G) will denote the Banach space of continuous complex-valued functions on G and M(G) ⊆ C*(G) the set of Borel probability measures on G. For μ, ν ∈ M(G), the convolution μ * ν is defined by

\[ \int_G f(x) d(\mu * \nu)(x) = \int_G \int_G f(x + y) d\mu(x) d\nu(y) \quad (f \in C(G)). \]
Now if $H$ is a subgroup of $G$, $H$ is open and closed if and only if the index $(G : H)$ of $H$ in $G$ is finite; if $G$ is zero-dimensional, this will be true whenever $H$ is the kernel of a continuous character on $G$. Let $H$ be an open-closed subgroup, and let $\mu \in M(G)$. We define $\mu^H \in M(G/H)$ as follows:

$$\mu^H([x]) = \mu([x]) \quad ([x] \in G/H).$$

Then it may be verified that for $\mu, \nu \in M(G)$,

$$(\mu \ast \nu)^H = \mu^H \ast \nu^H.$$  

We denote by $m_H(\mu)$ the maximum of the numbers $\mu^H([x])$.

**Theorem.** Let $G$ be a compact, zero-dimensional, Abelian group, and let $\mu_1, \mu_2, \ldots \in M(G)$. Then $\lim_{m \to \infty} \mu_{k,m} = \lambda_G$ in the weak-star topology for every $k$ if and only if

$$(2) \text{ for every proper open-closed subgroup } H \text{ of } G, \prod_{t=1}^{\infty} m_H(\mu_t) = 0.$$

**Remark.** This result, stated in a different form and without reference to convolutions, is obtained in Lemma 3 of [2] in the case where $G$ is the group of integers modulo two. In this case there is a simple proof involving the formula

$$k+m \sum_{i=k}^{\infty} |\mu_{k,m}(i)|.$$  

Our proof uses two preparatory lemmas.

**Lemma 1.** For any compact, Abelian group $G$ and any sequence $\mu_1, \mu_2, \ldots \in M(G)$, $\mu_{k,m}$ converges to $\lambda_G$ in the weak-star topology for every $k$ if and only if $\prod_{t=1}^{\infty} |\hat{\mu}_t(\gamma)| = 0$ for each $k$ and each nontrivial character $\gamma$ on $G$.

**Proof.** It is well known that a sequence $v_n$ in $M(G)$ converges to $v \in M(G)$ in the weak-star topology if and only if $v_n$ converges pointwise to $v$. We have $\hat{\lambda}_G(1) = 1, \hat{\mu}_t(1) = 1$ for each $t$, and $\hat{\lambda}_G(\gamma) = 0$ for $\gamma \neq 1$. The lemma follows from the fact that $\hat{\mu}_{k,m} = \hat{\mu}_k \hat{\mu}_{k+1} \cdots \hat{\mu}_{k+m}$.

**Lemma 2.** Suppose $r_i \geq 0$ ($i = 0, 1, \ldots, j-1$) and $\sum_{i=0}^{j-1} r_i = 1$. Let $R = \max r_i$, and denote by $w$ the $j$th root of unity $e(1/j) = \exp(2\pi i/j)$. Then

$$\left| \sum_{i=0}^{j-1} r_i w^i \right| \leq |R + (1 - R)w|.$$ 

The proof is straightforward.

**Proof of Theorem.** Suppose (2) holds. (In fact, we need only assume that it holds for subgroups $H$ which are kernels of continuous characters on $G$.) Let $\gamma$ be any continuous character other than 1; let $\gamma(G) = \{1, w, \ldots, w^{j-1}\}$ (where $w = e(1/j)$), and let $H = \ker \gamma$. Then for $t \geq 1$, the theorem follows from the fact that $\mu_{k,m} = \hat{\mu}_k \hat{\mu}_{k+1} \cdots \hat{\mu}_{k+m}$.
\[ \hat{\mu}_t(\gamma) = \int_G \overline{\gamma(x)} \, d\mu_t(x) = \sum_{i=0}^{j-1} r_{t,i} w^i, \]

where \( r_{t,i} = \mu_t(\gamma^{-1}(w^{-i})) \). Let \( R_t = \max_i r_{t,i} = m_H(\mu_t) \). Then from Lemma 2,

\[ |\hat{\mu}_t(\gamma)| \leq |R_t + (1 - R_t)w| \leq (1 - 2R_t(1 - R_t)(1 - \cos(2\pi/j)))^{1/2}. \]

By Lemma 1, it is sufficient to show that

\[ 3(1 - 2R_t(1 - R_t)(1 - \cos(2\pi/j))) = 0 \quad (k > 0), \]

which will be true if \( \sum_{i=1}^{\infty} R_t(1 - R_t) \) diverges. But since \( R_t \geq 1/j \), the conclusion follows from (2).

The proof in the other direction holds for any compact Abelian group and any closed subgroup \( H \) and is a straightforward application of (1). Q.E.D.

A few observations on the theorem are in order. First, it is easy to see that in the special case where \( \mu_1 = \mu_2 = \cdots = \mu \), condition (2) reduces to the statement in the theorem of Kawada and Itô at the beginning of the paper.

We consider next the special case in which

\[ df_t = f_t d\lambda_G, \]

where \( f_t \in L^2(G, \lambda_G) \), \( f_t \geq 0 \), and \( \int_G f_t d\lambda_G = 1 \). Then since \( f_t \in L^2 \), each of the sequences \( f_k * f_{k+1}, f_k * f_{k+1} * f_{k+2}, \ldots \) is equicontinuous and therefore has a uniformly convergent subsequence. It follows that each of these sequences converges uniformly to 1 if and only if, for every proper open-closed subgroup \( H \), \( \prod_{i=1}^{\infty} m_H(f_i) = 0 \) (where \( m_H(f_i) = m_H(\mu_t) \)).

Finally, we note that if (2) holds for any compact Abelian group \( G \), then \( \mu_{k,m}(f) \) converges to \( \lambda_G(f) \) for each \( k \), provided that \( f \) is a continuous function which is constant on cosets of some open-closed subgroup; zero-dimensionality assures that all trigonometric polynomials have this property.

One can consider the same question if the group \( G \) is non-Abelian. In this case, the statement of the theorem will require at least some modification. Clearly, a necessary condition for \( \mu_{k,m} \) to converge to \( \lambda_G \) for each \( k \) is that \( \prod_{i=1}^{\infty} \mu_t(H) = 0 \) for every proper open-closed subgroup \( H \); hence, the condition that (2) hold for every proper, normal, open-closed subgroup is not sufficient (though it is necessary). On the other hand, condition (2) as stated is not in general necessary: if \( G \) is the permutation group \( S_3 \), \( H = \{e, \phi\} \) is a subgroup of order 2, \( \psi \not\in H \), and \( \mu(\psi) = \mu(\psi\phi) = \frac{1}{2} \), we see that \( \mu^{(a)} \) converges to \( \lambda_G \) although the measure of the left coset \( \psi H \) is 1.

REFERENCES


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