FIXED POINTS AND ITERATION OF A NONEXPANSIVE MAPPING IN A BANACH SPACE

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Abstract. The following result is shown. If T is a nonexpansive mapping from a closed convex subset D of a Banach space into a compact subset of D and x₀ is any point in D, then the sequence \( \{x_n\} \) defined by \( x_{n+1} = 2^{-1}(x_n + Tx_n) \) converges to a fixed point of T. As a matter of fact, a theorem which includes this result is proved. Furthermore, a similar result is obtained under certain restrictions which do not imply the assumption on the compactness of T.

Throughout this paper we consider the following iterative procedure, which is a special case of the generalized iteration method introduced by W. R. Mann [7].

Definition. If D is a subset of a Banach space X, T is a mapping from D into X, and \( x_0 \in D \), then \( M(x_0, t_n, T) \) is the sequence \( \{x_n\}_{n=1}^{\infty} \) defined by \( x_{n+1} = (1 - t_n)x_n + t_nTx_n \), where \( \{t_n\}_{n=1}^{\infty} \) is a real sequence. If a point \( x_0 \) and a sequence \( \{t_n\}_{n=1}^{\infty} \) satisfy the following three conditions:

1. \( \sum_{n=1}^{\infty} t_n = \infty \),
2. \( 0 < t_n < b < 1 \) for all positive integers \( n \),

and

\( x_n \in D \) for all positive integers \( n \),

then \( x_0 \) and \( \{t_n\}_{n=1}^{\infty} \) will be said to satisfy Condition A.

Note that if \( t_n \in [a, b] \) for all positive integers \( n \) and \( 0 < a < b < 1 \), then it is obvious that the sequence \( \{t_n\}_{n=1}^{\infty} \) satisfies (1) and (2).

These iteration methods have been investigated by Krasnosel'skiĭ [6], Edelstein [3], Outlaw [9], Dotson [2] and others. They showed that these iterative methods may be used to find a fixed point of a nonexpansive mapping \( T \) mainly in a uniformly convex Banach space or a strictly convex Banach space, where a mapping \( T \) from a subset \( D \) of a Banach space \( X \) into \( X \) is called a nonexpansive mapping if \( T \) satisfies the condition that \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in D \).

In this paper we study the iterative method for nonexpansive mappings...
without any assumption on convexity of the Banach space.

**Lemma 1.** Let \( \{s_i\}_{i=1}^{\infty} \) be a sequence in the real numbers and let \( \{u_i\}_{i=1}^{\infty} \) be a sequence in a Banach space \( X \). Then for any positive integer \( N \),

\[
\left( \prod_{i=1}^{N-1} s_i \right) \left( \sum_{i=1}^{N} (1 - s_i)u_i \right)
\]

(3)

\[
= \left( 1 - \prod_{i=1}^{N} s_i \right) u_N - \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^{i} s_j \right) (u_{i+1} - s_i u_i) \right\}.
\]

If \( X \) is the real line and \( u_i = 1 \) for all \( i \), we have the special case

\[
\left( \prod_{i=1}^{N-1} s_i \right) \left( \sum_{i=1}^{N} (1 - s_i) \right)
\]

(4)

\[
= 1 - \prod_{i=1}^{N} s_i - \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^{i} s_j \right) (1 - s_i) \right\}.
\]

Here and hereafter we agree that \( \sum_{i=m}^{n} \) and \( \prod_{i=m}^{n} \) are defined to be 0 and 1, respectively, for \( n < m \).

**Proof.** When \( N = 1 \), the result is trivial. Supposing that (3) is true for some \( N > 1 \), we have

\[
\sum_{i=1}^{N} \left\{ \left( \prod_{j=i+1}^{N} s_j \right) \left( 1 - \prod_{j=1}^{i} s_j \right) (u_{i+1} - s_i u_i) \right\}
\]

\[
= s_N \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^{i} s_j \right) (u_{i+1} - s_i u_i) \right\}
\]

\[
- s_N \left( 1 - \prod_{i=1}^{N} s_i \right) u_N + \left( 1 - \prod_{i=1}^{N} s_i \right) u_{N+1}
\]

\[
= s_N \left\{ \left( 1 - \prod_{i=1}^{N} s_i \right) u_N - \left( \prod_{i=1}^{N-1} s_i \right) \left( \sum_{i=1}^{N} (1 - s_i) u_i \right) \right\}
\]

\[
- s_N \left( 1 - \prod_{i=1}^{N} s_i \right) u_N + \left( 1 - \prod_{i=1}^{N} s_i \right) u_{N+1}
\]

\[
= - \left( \prod_{i=1}^{N} s_i \right) \left( \sum_{i=1}^{N} (1 - s_i) u_i \right) + \left( 1 - \prod_{i=1}^{N} s_i \right) u_{N+1},
\]

from which it follows that

\[
\text{the right-hand side of (3) with } N + 1 \text{ for } N
\]

\[
= \left( 1 - \prod_{i=1}^{N+1} s_i \right) u_{N+1} - \sum_{i=1}^{N} \left\{ \left( \prod_{j=i+1}^{N} s_j \right) \left( 1 - \prod_{j=1}^{i} s_j \right) (u_{i+1} - s_i u_i) \right\}
\]

\[
= \left( 1 - s_{N+1} \prod_{i=1}^{N} s_i \right) u_{N+1} + \left( \prod_{i=1}^{N} s_i \right) \left( \sum_{i=1}^{N} (1 - s_i) u_i \right) - \left( 1 - \prod_{i=1}^{N} s_i \right) u_{N+1}
\]

\[
= \left( \prod_{i=1}^{N} s_i \right) \left( \sum_{i=1}^{N+1} (1 - s_i) u_i \right).
\]
By induction this completes the proof.

**Lemma 2.** Let $D$ be a subset of a Banach space $X$ and let $T$ be a nonexpansive mapping from $D$ into $X$. If there exist $x_1$ and $\{t_n\}_{n=1}^\infty$ that satisfy Condition A and $M(x_1, t_n, T)$ is bounded, then $x_n - Tx_n$ converges to zero as $n \to \infty$.

**Proof.** Since $T$ is a nonexpansive mapping, we have

$$
\|x_{n+1} - Tx_{n+1}\| = \|(1 - t_n)x_n + t_n Tx_n - Tx_{n+1}\|
$$

$$
= \|(1 - t_n)(x_n - Tx_n) + Tx_n - Tx_{n+1}\|
= (1 - t_n)\|x_n - Tx_n\| + \|x_n - x_{n+1}\|
= (1 - t_n)\|x_n - Tx_n\| + \|x_n - ((1 - t_n)x_n + t_n Tx_n)\|
= \|x_n - Tx_n\|.
$$

Thus the sequence $\{\|x_n - Tx_n\|\}_{n=1}^\infty$ is nonincreasing and bounded below, so $\lim_{n \to \infty} \|x_n - Tx_n\|$ exists.

Suppose that $\lim_{n \to \infty} \|x_n - Tx_n\| = r > 0$. That is, for any $\varepsilon > 0$, there exists an integer $m$ such that

$$
(5) \quad r < \|x_{m+i} - Tx_{m+i}\| \leq (1 + \varepsilon)r \quad \text{for all positive integers } i.
$$

Then since $T$ is nonexpansive,

$$
\|(Tx_{m+i+1} - x_{m+i+1}) - (1 - t_{m+i})(Tx_{m+i} - x_{m+i})\|
= \|T((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i})
- ((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - (1 - t_{m+i})(Tx_{m+i} - x_{m+i})\|
= \|T((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - Tx_{m+i}\|
\leq t_{m+i}\|x_{m+i} - Tx_{m+i}\| \leq t_{m+i}(1 + \varepsilon)r.
$$

Since $\{x_n\}_{n=1}^\infty$ is bounded and $\{t_n\}_{n=1}^\infty$ satisfies condition (1), there exists an integer $N$ such that

$$
(7) \quad r \sum_{i=1}^{N-1} t_{m+i} \leq \delta(M) + 1 \leq r \sum_{i=1}^N t_{m+i}
$$

where $\delta(M)$ is defined by $\sup\{\|x_i - x_j\|; 0 < i,j < \infty\}$.

Now setting $s_i = 1 - t_{m+i}$ and $u_i = Tx_{m+i} - x_{m+i}$ for all positive integers $i$, we get from (6),

$$
(8) \quad \|u_{i+1} - s_i u_i\| = \|Tx_{m+i+1} - x_{m+i+1} - (1 - t_{m+i})(Tx_{m+i} - x_{m+i})\|
\leq t_{m+i}(1 + \varepsilon)r = (1 - s_i)(1 + \varepsilon)r
$$

and

$$
x_{m+N+1} - x_{m+1} = \sum_{i=1}^N (((1 - t_{m+i})x_{m+i} + t_{m+i}Tx_{m+i}) - x_{m+i})
= \sum_{i=1}^N t_{m+i}(Tx_{m+i} - x_{m+i}) = \sum_{i=1}^N (1 - s_i)u_i.
$$
Thus using Lemma 1, we have from (9), (3), (5) and (8) that
\[
\left( \prod_{i=1}^{N-1} s_i \right) \| x_{m+N+1} - x_{m+1} \|
= \left\| \left( \prod_{i=1}^{N-1} s_i \right) \left( \sum_{i=1}^N (1 - s_i) u_i \right) \right\|
\geq \left( 1 - \prod_{i=1}^N s_i \right) \| u_N \| - \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^i s_j \right) \| u_{i+1} - s_j u_i \| \right\}
\geq \left( 1 - \prod_{i=1}^N s_i \right) r - \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^i s_j \right) (1 - s_i)(1 + \epsilon)r \right\}
= \left[ 1 - \prod_{i=1}^N s_i - \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^i s_j \right) (1 - s_i) \right\} \right] r
- \epsilon r \sum_{i=1}^{N-1} \left\{ \left( \prod_{j=i+1}^{N-1} s_j \right) \left( 1 - \prod_{j=1}^i s_j \right) (1 - s_i) \right\}.
\]

since \( s_i = 1 - t_{m+i} \geq 1 - b > 0 \), which implies from (4) and (7) that
\[
\| x_{m+N+1} - x_{m+1} \| \geq r \sum_{i=1}^N (1 - s_i) - \epsilon r \left( \prod_{i=1}^{N-1} s_i \right)^{-1}
\times \left\{ 1 - \prod_{i=1}^N s_i - \left( \prod_{i=1}^{N-1} s_i \right) \left( \sum_{i=1}^N (1 - s_i) \right) \right\}
\] 
\[
(10)
\geq r \sum_{i=1}^N (1 - s_i) - \epsilon r \left( \prod_{i=1}^{N-1} s_i \right)^{-1}
= r \sum_{i=1}^N t_{m+i} - \epsilon r \prod_{i=1}^{N-1} \left( 1 - t_{m+i} \right)^{-1}
\geq \delta(M) + 1 - \epsilon r \prod_{i=1}^{N-1} \left( 1 - t_{m+i} \right)^{-1}.
\]

Since \( \log(1 + y) \leq y \) for any \( y \in (-1, \infty) \), we have from (2) and (7),
\[
\prod_{i=1}^{N-1} \left( 1 - t_{m+i} \right)^{-1} = \prod_{i=1}^{N-1} \left( 1 + t_{m+i}(1 - t_{m+i})^{-1} \right)
= \exp \left\{ \sum_{i=1}^{N-1} \log \left( 1 + t_{m+i}(1 - t_{m+i})^{-1} \right) \right\}
\leq \exp \left\{ \sum_{i=1}^{N-1} t_{m+i}(1 - t_{m+i})^{-1} \right\}
\leq \exp \left\{ (1 - b)^{-1} \sum_{i=1}^{N-1} t_{m+i} \right\}
\leq \exp \left\{ (1 - b)^{-1} \left( \delta(M) + 1 \right) r^{-1} \right\}.
\]

From this and (10) we get that
\[
\delta(M) + 1 - \epsilon r \exp \left\{ (1 - b)^{-1} \left( \delta(M) + 1 \right) r^{-1} \right\}
\leq \| x_{m+N+1} - x_{m+1} \| \leq \delta(M).
\]

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Since $\epsilon$ is an arbitrary positive number, it follows that $\delta(M) + 1 \leq \delta(M)$. This contradiction completes the proof.

**Remark.** Let $T$ be a nonexpansive mapping from a convex set $D$ in a Banach space into a bounded subset of $D$ and let $(1 - t)I + iT$ be denoted by $T_t$, where $I$ is an identity map and $0 < t < 1$. Then $M(x_1, t, T)$ is bounded since it is a sequence in the convex hull of the union of $T(D)$ and the point $x_1$. Also it is clear that $T^n_t x_1 - T^{n-1}_t x_1 = t(Tx_n - x_n)$. Therefore we have by Lemma 2 that $T_t$ is asymptotically regular (i.e. for any $x \in D$, $\|T^{n+1}_t x - T^n_t x\| \to 0$ as $n \to \infty$).

**Fixed points and iterative process for compact mappings.** Now we shall prove a fixed point theorem for a nonexpansive compact mapping and show that the iterative process $M(x_1, t_n, T)$ may be used to find the fixed point.

**Theorem 1.** Let $D$ be a closed subset of a Banach space $X$ and let $T$ be a nonexpansive mapping from $D$ into a compact subset of $X$. If there exist $x_1$ and $\{t_n\}_{n=1}^{\infty}$ that satisfy Condition A, then $T$ has a fixed point in $D$ and $M(x_1, t_n, T)$ converges to a fixed point of $T$.

**Proof.** Let $D_0$ denote the closure of the convex hull of the union of $T(D)$ and the point $x_1$. A well-known theorem of Mazur implies that $D_0$ is compact. The sequence $M(x_1, t_n, T)$ clearly belongs to $D_0$. From this and Condition A, it immediately follows that $\{x_n\}_{n=1}^{\infty}$ is a compact sequence in $D$. Hence there is a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ that converges to a point $u$, which obviously belongs to $D$ since $D$ is closed. And it is clear that $\lim_{i \to \infty} \|Tx_{n_i} - x_{n_i}\| = 0$ since Lemma 2 is applicable from the boundedness of $D_0$.

Now since $T$ is nonexpansive,

$$\|Tu - u\| = \|Tu - Tx_{n_i} + Tx_{n_i} - x_{n_i} + x_{n_i} - u\| \\
\leq 2\|u - x_{n_i}\| + \|Tx_{n_i} - x_{n_i}\|,$$

which implies that $u$ is a fixed point of $T$ since $\lim_{i \to \infty} \|u - x_{n_i}\| = 0$ and $\lim_{i \to \infty} \|Tx_{n_i} - x_{n_i}\| = 0$.

Further,

$$\|x_{n+1} - u\| = \|(1 - t_n)x_n + t_n Tx_n - u\|$$

$$= \|(1 - t_n)(x_n - u) + t_n(Tx_n - Tu)\| \leq \|x_n - u\|$$

for any positive integer $n$. For any $\epsilon > 0$ there exists an integer $n_0$ such that $\|x_{n_0} - u\| < \epsilon$, so we obtain from (11) that $\|x_n - u\| < \epsilon$ for any integer $n \geq n_0$. Therefore $M(x_1, t_n, T)$ converges to $u$, a fixed point of $T$.

As an immediate consequence of Theorem 1, we have the following corollaries.

**Corollary 1.** Let $D$ be a closed subset of a Banach space $X$ and let $T$ be a nonexpansive mapping from $D$ into a compact subset of $X$. If there exists $t \in (0, 1)$ such that $(1 - t)x + tx \in D$ for all $x \in D$, then $T$ has a fixed point in $D$ and for any $x_1 \in D$, $M(x_1, t, T)$ converges to a fixed point of $T$.

**Corollary 2.** Let $D$ be a closed convex subset of a Banach space $X$ and let $T$
be a nonexpansive mapping from $D$ into a compact subset of $D$. Then $T$ has a fixed point in $D$ and $M(x_1, 2^{-1}, T)$ converges to a fixed point of $T$ for any $x_1 \in D$.

Note that the first part of Corollary 2 is a special case of a fixed point theorem of Schauder.

Corollary 2 was proved for uniformly convex spaces by Krasnosel’skiǐ [6] and strictly convex spaces by Edelstein [3].

**Fixed points and iterative process for noncompact mappings.** Next we shall consider the iterative process for a nonexpansive mapping without the assumption on the compactness of $T$.

Let $D$ be a subset of a Banach space $X$. A mapping $T: D \to X$ with a nonempty fixed points set $F$ in $D$ will be said to satisfy Condition B if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$, such that $\|x - Tx\| \geq f(d(x, F))$ for all $x \in D$, where $d(x, F) = \inf\{|x - z|; z \in F\}$. This condition is due to Senter and Dotson [10].

**Theorem 2.** Let $D$ be a closed subset of a Banach space $X$ and let $T: D \to X$ be a nonexpansive mapping with a nonempty fixed points set $F$ in $D$. If $T$ satisfies Condition B and there exist $x_1$ and $\{t_n\}_{n=1}^{\infty}$ that satisfy Condition A, then $M(x_1, t_n, T)$ converges to a member of $F$.

**Proof.** The theorem is trivial if $x_1 \in F$, so we assume $x_1 \in D - F$. For any $u \in F$ we have that $\|Tx_n - u\| \leq \|x_n - u\|$ and so we get that

$$\|x_{n+1} - u\| = \|(1 - t_n)x_n + t_nTx_n - u\| \leq \|x_n - u\|$$

which implies that $d(x_{n+1}, F) \leq d(x_n, F)$ for all positive integers $n$. The sequence $(d(x_n, F))_{n=1}^{\infty}$ is nonincreasing and bounded below, so there exists $\lim_{n \to \infty} d(x_n, F)$, which we denote by $r$.

By the definition of $f$, we have

$$\|x_n - Tx_n\| \geq f(d(x_n, F)) \geq f(r).$$

Since it follows from (12) that $M(x_1, t_n, T)$ is a bounded sequence in $D$, we have from Lemma 2 and (13) that $f(r) = 0$. Hence we get that

$$\lim_{n \to \infty} d(x_n, F) = r = 0.$$
which implies \( \{u_i\}_{i=1}^{\infty} \) is a Cauchy sequence, so there exists \( v \) such that \( v = \lim_{i \to \infty} u_i \) and \( v \) belongs to \( F \) since \( F \) is closed. For any \( \varepsilon > 0 \) there exists \( i_0 > 0 \) such that \( 2^{-i_0} < 2^{-1} \varepsilon \) and \( \|u_i_0 - v\| < 2^{-1} \varepsilon \), so we have that

\[
\|x_n - v\| \leq \|x_n - u_{i_0}\| + \|u_{i_0} - v\| < 2^{-i_0} + \|u_{i_0} - v\| < \varepsilon \text{ for all } n > i_0.
\]

Therefore \( M(x_1, t_n, T) \) converges to the point \( v \) of \( F \).

**Corollary 3.** Let \( D \) be a closed convex subset of \( X \) and let \( T: D \to D \) be a nonexpansive mapping with a nonempty fixed points set \( F \). If \( T \) satisfies Condition B, then for any \( x_1 \in D \) and any \( \{t_n\}_{n=1}^{\infty} \) satisfying (1) and (2), \( M(x_1, t_n, T) \) converges to a member of \( F \).

If \( X \) is a uniformly convex Banach space and \( 0 < a \leq t_n \leq b < 1 \) for all integers \( n > 0 \), the analog of this corollary was obtained by Senter and Dotson [10].

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**References**


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