A CHARACTERIZATION FOR THE PRODUCTS OF
k-AND- $\kappa_0$-SPACES AND RELATED RESULTS

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Abstract. E. Michael introduced the notion of $\kappa_0$-spaces and characterized spaces which are both $k$-spaces and $\kappa_0$-spaces (or, briefly, $k$-and-$\kappa_0$-spaces) as being precisely the quotient images of separable metric spaces.

The purpose of this paper is to give a necessary and sufficient condition for the product of two $k$-and-$\kappa_0$-spaces to be a $k$-and-$\kappa_0$-space. Moreover, as related matters, we shall consider the products of $k$-spaces having other properties.

1. Introduction. Throughout this paper, we shall assume that all spaces are regular, and all maps are continuous surjection.

According to E. Michael [7], an $\kappa_0$-space is a space with a countable pseudobase. Here a collection $\mathcal{P}$ of subsets of $X$ is a pseudobase for $X$ if, whenever $C \subseteq U$ with $C$ compact and $U$ open in $X$, then $C \subseteq P \subseteq U$ for some $P \in \mathcal{P}$.

In [7], it was proved that any countable product of $\kappa_0$-spaces is an $\kappa_0$-space. However, as is well known, the product of two $k$-and-$\kappa_0$-spaces need not be a $k$-space [4, Example 1.11].

As for the product of $k$-and-$\kappa_0$-spaces, we have the following main theorem. First, we state a definition.

K. Morita [11] introduced the notion of the class $\mathcal{E}'$. A space $X$ is of class $\mathcal{E}'$ if it is the union of countably many compact subsets $X_n$ such that $A \subseteq X$ is closed whenever $A \cap X_n$ is closed in $X_n$ for all $n$.

THEOREM 1.1. Let $X$ and $Y$ be $k$-and-$\kappa_0$-spaces. Then $X \times Y$ is a $k$-and-$\kappa_0$-space if and only if one of the following three properties holds:

1. $X$ and $Y$ are separable, metrizable spaces.
2. $X$ or $Y$ is a locally compact, separable metrizable space.
3. $X$ and $Y$ are spaces of class $\mathcal{E}'$.

As we shall see in §3, Theorem 1.1 can be extended to $\kappa$-spaces, a generalization of $\kappa_0$-spaces, which were introduced by P. O'Meara [17]. However, it cannot be extended to cosmic spaces in the sense of E. Michael [7]. As related matters, in §4, first, we shall give a characterization for the product of closed $s$-images of metric spaces to be a $k$-space. Second, we shall
consider conditions for the product of spaces having other properties to be a k-space.

2. Preliminary lemmas. Following P. O'Meara [17], a collection $\mathcal{T}$ of subsets of a space $X$ is a $k$-network for $X$ if, whenever $C \subset U$ with $C$ compact and $U$ open in $X$, then $C \subset \bigcup \{F; F \in \mathcal{T}\} \subset U$ for some finite subcollection $\mathcal{F}$ of $\mathcal{T}$.

An $\aleph$-space, according to O'Meara, is a space with a $\sigma$-locally finite $k$-network.

Clearly, metrizable spaces and $\aleph_0$-spaces are $\aleph$-spaces.

According to F. Siwiec [18], a space $X$ is strongly Fréchet (= countably bisequential in the sense of E. Michael [9]) if, whenever $\{F_n; n = 1, 2, \ldots\}$ is a decreasing sequence accumulating at $x$ in $X$, there exist $x_n \in F_n$ such that the sequence $\{x_n; n = 1, 2, \ldots\}$ converges to the point $x$.

Clearly, first-countable spaces are strongly Fréchet.

In [16], O'Meara proved that a first-countable $\aleph$-space is metrizable. We now show that this remains valid for a strongly Fréchet space.

**Lemma 2.1.** Let $X$ be a strongly Fréchet $\aleph$-space. Then $X$ is metrizable.

**Proof.** By the theorem in [16] quoted above, we need only prove $X$ is first-countable.

Let $\mathcal{T}$ be a $\sigma$-locally finite (or merely point-countable) closed $k$-network for $X$. Suppose $x \in X$. Let $\mathcal{T}' = \{P \in \mathcal{T}; x \in P\}$ and let $\mathcal{F}$ be the collection of all finite union of elements of $\mathcal{T}'$. Then the collection $\{\text{int } F; F \in \mathcal{F}\}$ is a countable local base for $x$.

Indeed, for each open subset $U$ containing $x$, let $\mathcal{T}'' = \{P \in \mathcal{T}' ; x \in P \subset U\}$. Then $\mathcal{T}''$ is a nonempty, countable subcollection of $\mathcal{T}'$. Let $\mathcal{P}'' = \{P_1, P_2, \ldots\}$. Let us show that $x \in \text{int } \bigcup P_n$ for some $n$.

In fact, suppose that $x \notin \text{int } \bigcup P_n$ for each $n$. Let $F_n = X - \bigcup P_n$. Then $\{F_n; n = 1, 2, \ldots\}$ is a decreasing sequence accumulating at $x$. Since $X$ is strongly Fréchet, there exist $x_n \in F_n$ such that the sequence $\{x_n; n = 1, 2, \ldots\}$ converges to the point $x$. The open subset $U$ contains $\{x_n; n \geq m\} \subset \{x\}$ for some $m$. Let $K = \{x_n; n \geq m\} \cup \{x\}$. Then there is a finite subcollection $\mathcal{G}$ of $\mathcal{T}$ such that $K \subset \bigcup \{F; F \in \mathcal{G}\} \subset U$. Some element $F$ of $\mathcal{G}$ contains the point $x$. Then $F \in \mathcal{T}''$. Since $F$ is a closed subset of $X$, there exists a subsequence of $K$ in $F$. This is a contradiction to the choice of the sequence $\{x_n; n = 1, 2, \ldots\}$.

K. Morita [12] introduced the notion of $M$-spaces and characterized paracompact $M$-spaces as being precisely the perfect inverse images of metric spaces.

**Lemma 2.2.** Let $X$ be an $\aleph$-space. If every closed subset of $X$ which is a paracompact $M$-space is locally compact, then $X$ has a $\sigma$-locally finite $k$-network consisting of compact subsets.

**Proof.** Let $\mathcal{T} = \bigcup_{n=1}^\infty \mathcal{T}_n$ be a $\sigma$-locally finite $k$-network for $X$. We assume that each element of $\mathcal{T}$ is closed, and for each $n$, $\mathcal{T}_n \subset \mathcal{T}_{n+1}$, and $\mathcal{T}_n$ is closed under finite intersections; that is, $\mathcal{T}_n$ contains all intersections of finitely many members of $\mathcal{T}_n$. 

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Let $K$ be a compact subset of $X$. Let $\mathcal{P}' = \{ P \in \mathcal{P}; P \cap K \neq \emptyset \}$, and let $\mathcal{K}$ be the collection of finite unions of elements of $\mathcal{P}'$ which contain the subset $K$. Then $\mathcal{K}$ is a nonempty, countable collection. Let $\mathcal{K} = \{ P_1, P_2, \ldots \}$, and $K_n = \bigcap_{i=1}^{n} P_i$ for each $n$. Then, for each open subset $U$ containing $K$, there is $K_n$ with $K \subseteq K_n \subseteq U$. That is, the decreasing collection \{ $K_n$; $n = 1, 2, \ldots$ \} of closed subsets is a countable local base for $K$.

Suppose that each $K_n$ is not compact. Then each $K_n$ is not countably compact. (Indeed, let some $K_n$ be countably compact. Then $K_n$ is first-countable, for each point of $K_n$ is a $G_\delta$-set in $K_n$. Thus, by the theorem of [16], $K_n$ is metrizable. Then $K_n$ is compact. This is a contradiction.) By the assumption, there is a closed subset $F = K \cup \bigcup_{n=1}^{\infty} D_n$ of $X$, where $D_n$ is a countable, infinite discrete subset of $K_n$. Let $Y$ be the quotient space obtained from $F$ by identifying all points of $K$. Then $Y$ is a countable, metrizable space which is not locally compact. Since $F$ is the perfect inverse image of $Y$, it is a paracompact $M$-space which is not locally compact. This is a contradiction to the hypothesis in this lemma. Hence some $K_m$ is compact, which implies all $K_n (n \geq m)$ are compact.

On the other hand, by the conditions on the collection $\mathcal{P}$, each $K_n$ can be expressed as a union of finitely many elements of $\mathcal{P}$. Then, for each open subset $U$ containing $K$, there is a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ such that each element of $\mathcal{F}$ is compact, and $K \subseteq \bigcup \{ F; F \in \mathcal{F} \} \subseteq U$.

Hence, it follows that $\{ P \in \mathcal{P}; P$ is compact $\}$ is a $\sigma$-locally finite $k$-network for $X$ consisting of compact subsets, which completes the proof.

According to E. Michael [9, Theorem 7.3], a $k$-space in which every point is a $G_\delta$-set is sequential in the sense of S. P. Franklin [4]. Then a $k$-and-$\aleph_0$-space is sequential, for each point of an $\aleph_0$-space is a $G_\delta$-set. Hence, by the theorem in [9] and [20, Theorem 1.1], we have

**Lemma 2.3.** Let $X$ be a $k$-and-$\aleph_0$-space, and let $Y$ be first-countable. If $X \times Y$ is a $k$-space, then $X$ is strongly Fréchet or $Y$ is locally countably compact.

Now, for the later convenience, we shall introduce the class $\mathcal{C}'$ which is broader than the class $\mathcal{E}'$. A space $X$ is said to belong to the class $\mathcal{C}'$ if it is the union of countably many closed and locally compact subsets $X_n$ such that $A \cap X$ is closed whenever $A \cap X_n$ is closed in $X_n$ for all $n$.

A Lindelöf space of the class $\mathcal{C}'$ belongs to the class $\mathcal{E}'$. A space of the class $\mathcal{C}'$ is a $k$-space.

**Lemma 2.4.** Let $X$ be a $k$-space. If $X$ has a $\sigma$-locally finite $k$-network consisting of compact subsets, then $X$ belongs to the class $\mathcal{C}'$.

**Proof.** Let $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$ ($\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for all $n$) be a $\sigma$-locally finite $k$-network for $X$ consisting of compact subsets. Let $P_n = \bigcup \{ P; P \in \mathcal{P}_n \}$. Then each $P_n$ is closed and locally compact. Since $\mathcal{P}$ is a $k$-network for $X$, each compact subset $K$ of $X$ is a subset of some $P_n$. Hence, it is easily seen that $X$ belongs to the class $\mathcal{C}'$.

The following lemma is proved as in the proof of [10, Lemma 2.1], so we shall omit the proof.

**Lemma 2.5.** Let $X$ and $Y$ belong to the class $\mathcal{C}'$. Then $X \times Y$ also belongs to the class $\mathcal{C}'$, and hence $X \times Y$ is a $k$-space.
3. **Proof of the main theorem.** We shall prove the following generalization of the main theorem.

**Theorem 3.1.** Let $X$ and $Y$ be $k$-and-$\mathfrak{N}$-spaces. Then $X \times Y$ is a $k$-and-$\mathfrak{N}$-space if and only if one of the following three properties holds:

1. $X$ and $Y$ are metrizable spaces.
2. $X$ or $Y$ is a locally compact, metrizable space.
3. $X$ and $Y$ are spaces of the class $\Xi'$. 

**Proof.** The “if” part is proved by [3, Theorem 3.2], Lemma 2.5, and by the fact that any countable product of $\mathfrak{N}$-spaces is an $\mathfrak{N}$-space (this is proved as in [7, Proposition 6.1]). So we shall prove the “only if” part.

**Case 1.** $X$ and $Y$ are strongly Fréchet: By Lemma 2.1, $X$ and $Y$ are metrizable.

**Case 2.** $X$ is strongly Fréchet and $Y$ is not strongly Fréchet: By Lemma 2.1, $X$ is metrizable, hence it is locally compact by Lemma 2.3. Similarly, if $X$ is not strongly Fréchet and $Y$ is strongly Fréchet, then $Y$ is locally compact, metrizable.

**Case 3.** Neither $X$ nor $Y$ is strongly Fréchet: Suppose that $X$ contains a closed, paracompact $M$-space $F$ which is not locally compact. Then $F$ is the perfect inverse image of a metric space $Z$ which is not locally compact. By assumption, $X \times Y$ is a $k$-space, then so the closed subset $F \times Y$ is also a $k$-space. Then $Z \times Y$ is a $k$-space, for it is the perfect image (hence, the quotient image) of the $k$-space $F \times Y$. Since $Y$ is not strongly Fréchet, by Lemma 2.3, $Z$ is locally compact. This is a contradiction. Hence $X$ does not contain a closed, paracompact $M$-space which is not locally compact. Thus, by Lemma 2.2, $X$ has a $\sigma$-locally finite $k$-network consisting of compact subsets. Hence, by Lemma 2.4, $X$ belongs to the class $\Xi'$. Similarly, $Y$ belongs to the class $\Xi'$. That completes the proof.

The following example shows that Theorems 1.1 and 3.1 become false if “$\mathfrak{N}_0$-space” is weakened to “cosmic space”.

**Example 3.2.** Let $C$ be the compact subspace $[0, 1] \times \{0\}$ of the “butterfly space” $S$ of L. F. McAuley; that is, of the plane with the usual topology at points not on the $x$-axis and with “bow-tie” neighborhoods of points on the $x$-axis. Let $Y$ be the quotient space obtained from $S$ by identifying all points of $C$. Then the cosmic space $Y$ is neither locally compact nor first-countable [2, Remark 3.3]. Let $X = Y^\omega$, the product of countably many copies of $Y$. Then $X \times X$ is cosmic, and since it is the perfect image of the first-countable space $S^\omega$, it is a $k$-space. However, $X$ is not metrizable and moreover, as in the proof of [13, Theorem 1], it follows that $X$ cannot be expressed as a countable union of closed, locally compact subsets. Thus the cosmic $k$-space $X$ satisfies none of the three properties of Theorem 3.1.

4. **Some related results.** First, we shall consider products of closed $s$-images of metric spaces.

It appears to be unknown whether every closed $s$-image of a metric space is an $\mathfrak{N}$-space. However, we have the following lemma.

**Recall that a map $f: X \to Y$ is a $s$-map if each $f^{-1}(y)$ has a countable base.**
Lemma 4.1. Let $X$ be the image of a metric space $Z$ under a closed $s$-map $f$. If every closed, metrizable subset of $X$ is locally compact, then $X$ is an $\alpha$-space.

Proof. Let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a $\sigma$-locally finite base for $Z$. We can assume that $\mathcal{B}_n \subset \mathcal{B}_{n+1}$, and that $\mathcal{B}_n$ is closed under finite intersections for each $n = 1, 2, \ldots$. Let $\mathcal{F}_n = \{f(B); B \in \mathcal{B}_n\}$, and $\mathcal{C}_n = \{F \in \mathcal{F}_n; F$ is compact $\}$ for each $n = 1, 2, \ldots$. Then, as in the proof of Lemma 2.2, $\bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a $\sigma$-hereditarily closure-preserving closed network for $X$. Put $C_n = \bigcup\{C; C \in \mathcal{C}_n\}$.

We shall prove that each closed subset $C_n$ of $X$ is an $\alpha$-space. Let $x \in C_n$. Then, since $f$ is a closed and $s$-map, there is an open subset $V$ of $C_n$ containing $x$ which is covered by countably many elements $C_{n,i}$ of $\mathcal{C}_n$. Since the collection $\mathcal{V} = \{V \cap C_{n,i}; i = 1, 2, \ldots\}$ is a hereditarily closure-preserving closed covering of $V$, it follows that each compact subset of $V$ is covered by finitely many elements of $\mathcal{V}$. Also each element of $\mathcal{V}$ is compact, hence compact metric, and hence an $\alpha_0$-space. Thus it is easily checked that $V$ is an $\alpha_0$-space, and hence so is $V$. This implies that $C_n$ is locally an $\alpha_0$-space. On the other hand, $C_n$ is a closed image of a metric space. Then, by [5], $C_n$ is paracompact. Thus $C_n$ is paracompact and locally an $\alpha_0$-space. Hence it follows that each $C_n$ is an $\alpha$-space.

Now $X$ is the closed image of a metric space, so by [6, Corollary 1.2] each compact subset of $X$ is the image of some compact subset of $Z$. On the other hand, $\{B \in \mathcal{B}; f(B) \in \bigcup_{n=1}^{\infty} \mathcal{C}_n\}$ is an open covering of $Z$. Hence it follows that each compact subset of $X$ is covered by finitely many closed $\alpha$-spaces $C_n$. This implies that $X$ is an $\alpha$-space.

Lemma 4.2 [9, Corollary 9.10]. Let $X$ be the closed image of a metric space. If $X$ is strongly Frechet, then it is metrizable.

From the proof of Theorem 3.1, and from Lemmas 4.1 and 4.2, we have

Theorem 4.3. Let $X$ and $Y$ be the closed $s$-images of metric spaces. Then $X \times Y$ is a $k$-space if and only if one of the three properties of Theorem 3.1 holds.

The question, however, remains whether this theorem is also valid with "$s$-image" weakened to "image".

Finally, we shall show that the proofs in §3 also give more information about the products of $k$-spaces having other properties. In fact, we have the following Theorems 4.4 and 4.5. The latter is a generalization of [20, Theorem 1.1].

Theorem 4.4 and the "only if" part of Theorem 4.5 are proved as in the proofs of Theorem 3.1 and Lemma 2.2, with Lemma 2.3 replaced by [20, Theorem 1.1]. As for Theorem 4.4, we make use of [15, Theorem 3.6], [14, Lemma 1.4], and the fact that every countably compact, strong $\Sigma$-space is compact (this is easily proved).

Theorem 4.4. Let $X$ be a Frechet space, or a $k$-space in which every point is a $G_\delta$-set. Let $Y$ be a $\sigma$-space (resp. a strong $\Sigma$-space in the sense of K. Nagami [14]). If $X \times Y$ is a $k$-space, then $X$ is strongly Frechet, or $Y$ is the countable
union of closed, locally compact, metric (resp. locally compact, paracompact) subsets.

**Theorem 4.5.** Let $X$ have the same properties as in Theorem 4.4. Let $Y$ be a space of pointwise countable type in the sense of A. V. Arhangel’$^\prime$ski$\bar{i}$ [1, p. 37]. Then $X \times Y$ is a $k$-space if and only if $X$ is strongly Fréchet, or $Y$ is locally countably compact.

The "if" part of this theorem is proved by [19, Corollary 2.4] and Proposition 4.6 below.

For the definitions of bi-$k$-spaces and countably bi-$k$-spaces, see [9, Definitions 3.E.1 and 4.E.1]. Obviously, spaces of pointwise countable type are bi-$k$, and strongly Fréchet spaces are countably bi-$k$.

**Proposition 4.6.** Let $X$ be countably bi-$k$ and let $Y$ be bi-$k$. Then $X \times Y$ is a $k$-space.

**Proof.** Combining some known facts, this is proved step by step. By [9, Theorem 3.E.3], $Y$ is a biquotient image of a paracompact $M$-space $Z$. Then, by [8, Theorem 1.2], $X \times Y$ is a biquotient image of $X \times Z$. We shall prove that $X \times Z$ is a $k$-space. By [12, Theorem 6.1], $Z$ is a perfect inverse image of a metric space $T$. Thus $X \times Z$ is a perfect inverse image of $X \times T$. But $X \times T$ is a $k$-space by [9, Theorem 4.E.3]. Hence $X \times Z$ is a $k$-space by [1, Theorem 2.5]. Thus $X \times Y$ is a $k$-space, for it is the biquotient image (hence, the quotient image) of the $k$-space $X \times Z$.

**References**

8. K. Nagami, *Some generalizations of metric spaces, their metrization theorems and product


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