HOMEOMORPHISMS WITH MANY RECURRENT POINTS

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ABSTRACT. Let X be a topological space and H(X) the space of all homeomorphisms of X onto itself with the compact open topology. If \( f \in H(X) \) and \( p \in X \), then \( p \) is a recurrent point of \( f \) provided \( p \) is in the closure of \( \{ f^n(p) \mid n \geq 1 \} \). It is shown that if \( X \) is Hausdorff and \( V \) is a nonempty open subset of \( X \) homeomorphic to Euclidean \( n \)-dimensional space with \( n \geq 1 \), then \( \{ f \in H(X) \mid \text{the recurrent points of } f \text{ are dense in } V \} \) is nowhere dense in \( H(X) \).

Let \( X \) be a topological space and \( H(X) \) the space of all homeomorphisms of \( X \) onto itself with the compact open topology. If \( f \in H(X) \) and \( p \in X \), then \( p \) is a recurrent point of \( f \) provided \( p \in \overline{\{ f^n(p) \mid n \geq 1 \}} \), where \( \overline{A} \) denotes the closure of \( A \). Given a fixed Borel measure \( \mu \) on \( X \) which assigns a positive value to each nonempty open subset of \( X \), we say an \( f \in H(X) \) is recurrent (with respect to \( \mu \)) provided the set of nonrecurrent points of \( f \) has \( \mu \)-measure zero. The following theorem is proven in [1].

**Theorem 1.** If \( X \) is a compact manifold without boundary, of dimension greater than zero and of nonzero Euler characteristic, then the set of all recurrent homeomorphisms of \( X \) is nowhere dense in \( H(X) \).

We establish here a stronger conclusion than that of Theorem 1 while removing the hypotheses that \( X \) is compact and has nonzero Euler characteristic. We also weaken the hypothesis that \( X \) is an \( n \)-dimensional manifold without boundary, \( n \geq 1 \), to the assumption that there is a nonempty open subset of \( X \) homeomorphic to some Euclidean space \( \mathbb{R}^n \) with \( n \geq 1 \). The proof is a modification of the argument in [1].

**Theorem 2.** If \( X \) is a Hausdorff space and \( V \) is an open subset of \( X \) homeomorphic to \( \mathbb{R}^n \) with \( n \geq 1 \), then \( R = \{ f \in H(X) \mid \text{the recurrent points of } f \text{ are dense in } V \} \) is nowhere dense in \( H(X) \).

**Proof.** Assume the hypothesis and the contrary to the conclusion. Then there is a nonempty open set \( W \subset H(X) \) such that \( W \subset \overline{R} \). Let \( f \in W \cap R \). Then the recurrent points of \( f \) are dense in \( V \). Let \( p \) be one.

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Claim. There exists a $\varphi \in W$ such that $\varphi^k(p) = p$ for some $k \geq 1$.

Proof. Since $V$ is homeomorphic to $\mathbb{R}^n$, $n \geq 1$, we identify $V$ with $\mathbb{R}^n$. Let $\epsilon$ be any positive number. Since $p$ is a recurrent point for $f$, $f^k(p) \in U_\epsilon = \{ q \in V = \mathbb{R}^n \| q - p \| < \epsilon \}$ (where $\| \cdot \|$ is the Euclidean norm) for some $k \geq 1$. Let $k$ be the smallest such integer. Let $a$ be a homeomorphism of the closed ball $\overline{U}_\epsilon = \{ q \in V \| q - p \| \leq \epsilon \}$ onto itself such that $a$ is fixed on the sphere $S_\epsilon = \{ q \in V \| q - p \| = \epsilon \}$ and $a(f^k(p)) = p$. Extend $a$ to a homeomorphism of $X$ onto itself by setting $a(x) = x$ for $x \notin U_\epsilon$. Set $\varphi = a \circ f$. It is immediate that $\varphi^i(p) = f^i(p)$ for $0 \leq i < k$ and $\varphi^k(p) = p$. It is not hard to see that for $\epsilon$ sufficiently small, $\varphi \in W$. Indeed, for the special case of $W = N(C, O) = \{ h \in H(X) \| h(C) \subset O \}$, where $C$ is compact and $O$ is open, we have $\varphi \in W$ whenever

$$\epsilon < \min(1, \frac{1}{2} \text{ the distance from } f(C) \cap \overline{U}_1 \text{ to } \mathbb{R}^n - O \cap U_2).$$

The general case follows because the sets $N(C, O)$ generate the compact open topology. This proves the claim.

The rest of the proof follows the proof in [1].

Let $k$ be the smallest positive integer such that $\varphi^k(p) = p$. Pick an $\epsilon > 0$ such that $U_\epsilon, \varphi(U_\epsilon), \varphi^2(U_\epsilon), \ldots, \varphi^{k-1}(U_\epsilon)$ are pairwise disjoint. Pick a $\tau, 0 < \tau < \epsilon/3$ such that $\varphi^k(U_\tau) \subset U_{\epsilon/3}$. Let $\eta$ be a homeomorphism from $U_\tau$ onto itself such that $\eta$ is fixed on $S_\tau$ and $\eta(U_{2\epsilon/3}) \subset U_{\epsilon/2}$. See [1] for a detailed construction. Extend $\eta$ to a homeomorphism of $X$ onto itself by setting $\eta(x) = x$ for $x \notin U_\tau$. With an argument similar to that above it is not hard to see that for $\epsilon$ sufficiently small, $g = \varphi \circ \eta \in W$.

For $k \geq 2$ pick $t_i, i = 1, 2, \ldots, k$, such that $\frac{1}{2}\tau = t_1 < t_2 < \cdots < t_k = \tau$. For all $k \geq 1$ set $N_i = N(U_{2\epsilon/3}, \varphi(U_{\epsilon/2}))$ and

$$N_i = N(\varphi^{i-1}(U_{\epsilon/3}), \varphi^i(U_{\epsilon/3}))$$

for all $i, 2 \leq i \leq k$. Next, set $N = \bigcap_{i=1}^k N_i$. Unravelling the definition of the basic open set $N$, we see that if $h \in N$ and $q \in U_{2\epsilon/3} - U_{\epsilon/3}$ (which is nonempty because $n \geq 1$), then $q$ is not a recurrent point of $h$. This also uses the fact that $U_\epsilon, \varphi(U_\epsilon), \ldots, \varphi^{k-1}(U_\epsilon)$ are pairwise disjoint. This shows that $N \cap R = \emptyset$. But $g \in N$ and so $g \notin R$. Yet, $g \in W \subset R$, a contradiction. Q.E.D.

References


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