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HOMEOMORPHISMS WITH MANY RECURRENT POINTS

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ABSTRACT. Let X be a topological space and $H(X)$ the space of all homeomorphisms of X onto itself with the compact open topology. If $f \in H(X)$ and $p \in X$, then p is a recurrent point of f provided p is in the closure of $\{f^n(p) | n \geqslant 1\}$. It is shown that if X is Hausdorff and V is a nonempty open subset of X homeomorphic to Euclidean n-dimensional space with $n \geqslant 1$, then $\{f \in H(X) |$ the recurrent points of f are dense in $V\}$ is nowhere dense in $H(X)$.

Let X be a topological space and $H(X)$ the space of all homeomorphisms of X onto itself with the compact open topology. If $f \in H(X)$ and $p \in X$, then p is a recurrent point of f provided $p \in \{\overline{f^n(p) | n \geqslant 1}\}$, where $\overline{A}$ denotes the closure of A. Given a fixed Borel measure μ on X which assigns a positive value to each nonempty open subset of X, we say an $f \in H(X)$ is recurrent (with respect to μ) provided the set of nonrecurrent points of f has μ-measure zero. The following theorem is proven in [1].

THEOREM 1. *If X is a compact manifold without boundary, of dimension greater than zero and of nonzero Euler characteristic, then the set of all recurrent homeomorphisms of X is nowhere dense in $H(X)$.*

We establish here a stronger conclusion than that of Theorem 1 while removing the hypotheses that X is compact and has nonzero Euler characteristic. We also weaken the hypothesis that X is an n-dimensional manifold without boundary, $n \geqslant 1$, to the assumption that there is a nonempty open subset of X homeomorphic to some Euclidean space $\mathbf{R}_n$ with $n \geqslant 1$. The proof is a modification of the argument in [1].

THEOREM 2. *If X is a Hausdorff space and V is an open subset of X homeomorphic to $\mathbf{R}^n$ with $n \geqslant 1$, then $R = \{f \in H(X) |$ the recurrent points of f are dense in $V\}$ is nowhere dense in $H(X)$.*

PROOF. Assume the hypothesis and the contrary to the conclusion. Then there is a nonempty open set $W \subset H(X)$ such that $W \subset \overline{R}$. Let $f \in W \cap R$. Then the recurrent points of f are dense in V. Let p be one.

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Claim. There exists a $\varphi \in W$ such that $\varphi^k(p) = p$ for some $k \geqslant 1$.

PROOF. Since V is homeomorphic to $\mathbf{R}^n$, $n \geqslant 1$, we identify V with $\mathbf{R}^n$. Let ε be any positive number. Since p is a recurrent point for f, $f^k(p) \in U_\varepsilon = \{q \in V = \mathbf{R}^n \mid \|q - p\| < \varepsilon\}$ (where $\|\cdot\|$ is the Euclidean norm) for some $k \geqslant 1$. Let k be the smallest such integer. Let α be a homeomorphism of the closed ball $\overline{U}_\varepsilon = \{q \in V \mid \|q - p\| \leqslant \varepsilon\}$ onto itself such that α is fixed on the sphere $S_\varepsilon = \{q \in V \mid \|q - p\| = \varepsilon\}$ and $\alpha(f^k(p)) = p$. Extend α to a homeomorphism of X onto itself by setting $\alpha(x) = x$ for $x \notin U_\varepsilon$. Set $\varphi = \alpha \circ f$. It is immediate that $\varphi^i(p) = f^i(p)$ for $0 \leqslant i < k$ and $\varphi^k(p) = p$. It is not hard to see that for ε sufficiently small, $\varphi \in W$. Indeed, for the special case of $W = N(C, O) = \{h \in H(X) \mid h(C) \subset O\}$, where C is compact and O is open, we have $\varphi \in W$ whenever

$$\varepsilon < \min(1, \tfrac{1}{2} \text{ the distance from } f(C) \cap \overline{U}_1 \text{ to } \mathbf{R}^n - O \cap U_2).$$

The general case follows because the sets $N(C, O)$ generate the compact open topology. This proves the claim.

The rest of the proof follows the proof in [1].

Let k be the smallest positive integer such that $\varphi^k(p) = p$. Pick an $\varepsilon > 0$ such that $U_\varepsilon, \varphi(U_\varepsilon), \varphi^2(U_\varepsilon), \ldots, \varphi^{k-1}(U_\varepsilon)$ are pairwise disjoint. Pick a τ, $0 < \tau < \varepsilon/3$ such that $\varphi^k(U_\tau) \subset U_{\varepsilon/3}$. Let η be a homeomorphism from $\overline{U}_\varepsilon$ onto itself such that η is fixed on S_ε and $\eta(\overline{U}_{2\varepsilon/3}) \subset U_{\tau/2}$. See [1] for a detailed construction. Extend η to a homeomorphism of X onto itself by setting $\eta(x) = x$ for $x \notin U_\varepsilon$. With an argument similar to that above it is not hard to see that for ε sufficiently small, $g = \varphi \circ \eta \in W$.

For $k \geqslant 2$ pick t_i, $i = 1, 2, \ldots, k$, such that $\frac{1}{2}\tau = t_1 < t_2 < \cdots < t_k = \tau$. For all $k \geqslant 1$ set $N_1 = N(\overline{U}_{2\varepsilon/3}, \varphi(U_{\tau/2}))$ and

$$N_i = N(\varphi^{i-1}(\overline{U}_{t_{i-1}}), \varphi^i(U_{t_i}))$$

for all i, $2 \leqslant i \leqslant k$. Next, set $N = \bigcap_{i=1}^k N_i$. Unravelling the definition of the basic open set N, we see that if $h \in N$ and $q \in U_{2\varepsilon/3} - \overline{U}_{\varepsilon/3}$ (which is nonempty because $n \geqslant 1$), then q is not a recurrent point of h. This also uses the fact that $U_\varepsilon, \varphi(U_\varepsilon), \ldots, \varphi^{k-1}(U_\varepsilon)$ are pairwise disjoint. This shows that $N \cap R = \varnothing$. But $g \in N$ and so $g \notin \overline{R}$. Yet, $g \in W \subset \overline{R}$, a contradiction. Q.E.D.

REFERENCES

1. Chung-wu Ho, *On the homeomorphisms which satisfy the Poincaré Recurrence Theorem*, Proc. Amer. Math. Soc. **58** (1976), 272–276.

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