A SUBALGEBRA CONDITION IN LIE-ADMISSIBLE ALGEBRAS

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The author wishes to present this paper in memory of his wife, Myung Mi Myung, whose untimely death occurred during the preparation of this paper. She was a trained mathematician and unselfishly encouraged the author during her illness and assisted in the preparation of the manuscript.

Abstract. Let $A$ be a finite-dimensional, flexible, Lie-admissible algebra over a field $\phi$ of characteristic $\neq 2$. Let $S$ be a subalgebra of $A^-$ and $H$ be a Cartan subalgebra of $S$. It is shown that $S$ is a subalgebra of $A$ if and only if $HH \subseteq S$.

For an algebra $A$, denote by $A^-$ the algebra with multiplication $[x,y] = xy - yx$ defined on the vector space $A$. If $A^-$ is a Lie algebra then $A$ is said to be Lie-admissible. If $A$ is, in addition, a finite-dimensional flexible algebra over a field $\phi$ then a Cartan subalgebra of $A^-$ has played an important role for the structure of the algebra $A$ [1], [3]. Let $S$ be a subalgebra of the Lie algebra $A^-$. In this note, we give a condition in terms of a Cartan subalgebra of $S$ that $S$ be a subalgebra of $A$.

Theorem. Let $A$ be a finite-dimensional, flexible, Lie-admissible algebra over a field $\phi$ of characteristic $\neq 2$. Let $S$ be a subalgebra of the Lie algebra $A^-$ and $H$ be a Cartan subalgebra of $S$. Then $S$ is a subalgebra of $A$ if and only if $HH \subseteq S$.

Proof. One can assume that $\phi$ is algebraically closed. Since $A$ is flexible and Lie-admissible, the mapping $\text{ad} x: a \rightarrow [a,x]$ is a derivation of $A$ for all $x \in A$; that is, $[a,bc] = [a,b]c + b[a,c]$ for all $a, b \in A$. Let $x, y \in A$. Then the flexible law $(x,y,y) + (y,y,x) = 0$ implies $y(xy) = (xy)y - [x,y^2]$. Hence we get

$$[[x,y],y] = (xy)y - (yx)y - y(xy) + y(xy) = (xy)y - 2y(xy) + y(xy)$$

$$= 2(xy)y - [x,y^2] - 2y(xy) = 2[xy,y] - [x,y^2] = 2[xy,y] - [x,y^2].$$

Therefore, we have $[x,y]y = \frac{1}{2}([x,y^2] + [[x,y],y])$.

Let $S_n = \{ x \in S | x(\text{ad} h - a(h))^{n} = 0, h \in H, \text{ for some } n > 0 \}$. Then since $H$ is a Cartan subalgebra of $S$, we have the root space decomposition

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$S = \sum S_\alpha$ of $S$ relative to $H$, where $H_0 = S$ and $[S_\alpha, H] \subseteq S_\alpha$. For $\alpha \neq 0$, we first show that $S_\alpha H \subseteq S$. Since $\alpha \neq 0$, there exists an element $h \in H$ such that $\alpha(h) \neq 0$. Then $\text{ad} h : S_\alpha \to S_\alpha$ is surjective. For, if $[x, h] = 0$ for some $x \neq 0$ in $S_\alpha$ then $x(\text{ad} h - \alpha(h)I)^n = 0$ implies that $\alpha(h)^n x = 0$ and so $\alpha(h) = 0$. Now, for every element $x \in S_\alpha$, we have

$[x, h]_h = \frac{1}{2}([x, h^2] + [[x, h], h]) \in [S, HH] + [[S, H], H] \subseteq S.$

Since $[S_\alpha, h] = S_\alpha$, this implies $S_\alpha h \subseteq S$. Let $k$ be any element in $H$. Then we have

$[x, h]k = [x, hk] - h[x, k] = [x, hk] - [h, [x, k]] - [x, k]h$

$\in [S, HH] + [H, [S, H]] + [S_\alpha, H]h \subseteq S + S_\alpha h \subseteq S.$

Again, since $[S_\alpha, h] = S_\alpha$, this implies $S_\alpha H \subseteq S$ for $\alpha \neq 0$. Also, $S_0 H = HH \subseteq S$ and so we have that $SH \subseteq S$.

For any $\alpha \neq 0$, let $h$ be an element in $H$ such that $\alpha(h) \neq 0$. Let $x \in S_\alpha, y \in S$. Then

$y[x, h] = [x, yh] - [x, y]h \in [S, SH] + SH \subseteq S.$

Since $[S_\alpha, h] = S_\alpha$, this shows that $SS_\alpha \subseteq S$ for $\alpha \neq 0$. From $SS_0 = SH \subseteq S$, we have that $SS \subseteq S$, as required.

If $S$ is a subalgebra of $A^-$ which is classical in the sense of Seligman [4], then, in view of [3, Corollary 3.4], the theorem enables us to give a condition that $S$ is a Lie algebra under the multiplication in $A$, so that a classical Lie algebra is imbedded into $A$ as a subalgebra. An element $x \in A$ is called nilpotent if $x$ is power-associative and $x^n = 0$ for some $n > 0$. We also say that a subset $M$ of $A$ is nil if every element of $M$ is nilpotent. The following is an immediate consequence of the theorem and [3, Corollary 3.4].

**Corollary 1.** Let $S$ be a subalgebra of $A^-$ which is classical and $H$ be a classical Cartan subalgebra of $S$. Then $S$ is a Lie algebra under the multiplication in $A$ if and only if $HH \subseteq S$ and $H$ is nil in $A$.

In particular, if $A$ is power-associative and $A^-$ is semisimple over $\Phi$ of characteristic 0, it is shown that $A$ is a nilalgebra [2] and turns out to be a Lie algebra [1], [3]. The original proof of this requires that $\Phi$ is algebraically closed; however, if $\Phi$ is not algebraically closed, it can be extended to its algebraic closure. Therefore, we have

**Corollary 2.** Let $\Phi$ be of characteristic 0 and let $S$ be a semisimple subalgebra of $A^-$. Suppose that every element of $S$ is power-associative. Then $S$ is a Lie algebra under the multiplication in $A$ if and only if $S$ contains a Cartan subalgebra $H$ such that $HH \subseteq S$.

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