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REMARKS ON THE CARDINALITY OF COMPACT SPACES AND THEIR LINDELÖF SUBSPACES

A. HAJNAL AND I. JUHÁSZ

ABSTRACT. Several applications of the Čech-Pospišil theorem are given; one of them states (under CH) that every uncountable compact space has a Lindelöf subspace of cardinality ω_1.

Notation and definitions are the same as in [1], except that a G_κ set means a set which is the intersection of $\leqslant \kappa$ open sets. Thus if $\kappa = \omega_\xi$, then G_κ is the same as $G_{\delta,\xi}$ in [1].

Let us recall at the outset the old theorem of Čech and Pospišil which states that if X is compact T_2 and $\chi(p, X) \geqslant \kappa \geqslant \omega$ hold for each $p \in X$, then $|X| \geqslant 2^\kappa$. In [1] and [2] it is (incorrectly) stated that this theorem remains valid if we only require X to be a G_κ subset of some compact T_2 space. This is not true; however, it turns out that if we replace G_κ with G_λ for some $\lambda < \kappa$, then a simple argument can reduce it to the Čech-Pospišil theorem.

PROPOSITION 1. *If $X \subset Z$ is a (nonempty) G_λ-subset with $\lambda < \kappa$, where Z is compact Hausdorff, and $\chi(p, X) \geqslant \kappa$ for each $p \in X$, then $|X| \geqslant 2^\kappa$.*

PROOF. It is known (cf. [1, 2.11]) that X contains a nonempty subset Y which is a *closed* G_λ-subset of Z. Thus Y is compact and we claim that $\chi(p, Y) \geqslant \kappa$ for each $p \in Y$. Indeed, since $\psi(p, Y) = \chi(p, Y)$, otherwise we would have $\psi(p, Z) = \chi(p, Z) < \kappa$, considering that Y is a G_λ-subset of Z, contradicting that $\kappa \leqslant \chi(p, X) \leqslant \chi(p, Z)$. Hence we have $2^\kappa \leqslant |Y| \leqslant |X|$.

The following result is not proved directly from the Čech-Pospišil theorem but from its proof. It seems to be new and will be crucial in the proof of Theorem 1.

PROPOSITION 2. *If X is a compact Hausdorff space with $\chi(p, X) \geqslant \kappa$ for all $p \in X$, then there exists a subset $C \subset X$ with $|C| \leqslant 2^{\underline{\kappa}} = \Sigma\{2^\lambda \colon \lambda < \kappa\}$ such that $|\overline{C}| \geqslant 2^\kappa$.*

PROOF. The basic idea in all known proofs of the Čech-Pospišil theorem is the construction of a certain kind of ramification system of closed sets. To be more precise, for every 0-1 sequence ε of length $\nu \leqslant \kappa$ one defines a nonempty closed set $X_\varepsilon \subset X$ such that

(i) if ν is a limit ordinal, then $X_\varepsilon = \bigcap \{X_{\varepsilon \restriction \mu} \colon \mu < \nu\}$;

(ii) $X_{\varepsilon\frown 0}, X_{\varepsilon\frown 1} \subset X_\varepsilon$ and $X_{\varepsilon\frown 0} \cap X_{\varepsilon\frown 1} = \varnothing$.

Let S_ν denote the set of all 0-1 sequences of length ν. For every $\nu < \kappa$, $\varepsilon \in S_\nu$ and $i < 2$, the set $X_\varepsilon \setminus X_{\varepsilon\frown i}$ is nonempty; hence we can choose for each

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such ε and i a point $p_{\varepsilon,i} \in X_\varepsilon \setminus X_{\widehat{\varepsilon i}}$. Now put

$$C = \{p_{\varepsilon,i}: \varepsilon \in \bigcup\{S_\nu: \nu < \kappa\} \ \&\ i \in 2\}.$$

It is obvious that $|C| \leqslant 2^{\underline{\kappa}}$. Next we show that for each $\eta \in S_\kappa$ we have $X_\eta \cap \overline{C} \neq \emptyset$; this will obviously imply $|\overline{C}| \geqslant 2^\kappa$. To see this, let

$$C_\eta = \{p_{\eta\restriction\nu,\eta(\nu)}: \nu < \kappa\} \subset C.$$

(For simplicity we write $p_{\eta\restriction\nu,\eta(\nu)} = p^{(\nu)}$.) Then $\mu < \nu < \kappa$ implies $p^{(\mu)} \neq p^{(\nu)}$, hence $|C_\eta| = \kappa$. Since X is compact, C_η must have a complete accumulation point in X, and we claim that any such point must belong to X_η. This will imply $\overline{C} \cap X_\eta \supset \overline{C}_\eta \cap X_\eta \neq \emptyset$. Indeed, let $q \in X \setminus X_\eta$ arbitrary. Then there is a $\nu < \kappa$ such that $q \in X \setminus X_{\eta\restriction\nu}$, but obviously, this latter set is an open neighbourhood of q, whose intersection with C_η is $\{p^{(\mu)}: \mu < \nu\}$, hence of cardinality less than κ. This completes the proof.

The next result was obtained while trying to solve the following problem: Is it true that a Lindelöf space of cardinality ω_2 must contain a Lindelöf subspace of cardinality ω_1? (GCH is assumed.) A natural thing was to try it for compact spaces first.

THEOREM 1. *Assume the CH (i.e. $2^\omega = \omega_1$). Then every uncountable compact space has a Lindelöf subspace of cardinality ω_1.*

Before we can start the actual proof, we state a lemma, which is interesting in itself.

LEMMA. *If a space X has a point $p \in X$ with $\psi(p, X) = \chi(p, X) = \omega_1$, then X contains a Lindelöf subspace of cardinality ω_1.*

PROOF. Let $\{U_\nu: \nu < \omega_1\}$ be a basis of neighbourhoods for p in X. We define points p_ν for $\nu < \omega_1$ by transfinite induction, as follows. Suppose that $\nu < \omega_1$, and for each $\mu < \nu$ the point p_μ has already been defined.

Then we choose p_ν in $\bigcap\{U_\mu: \mu \leqslant \nu\} \setminus (\{p_\mu: \mu < \nu\} \cup \{p\})$, which is possible by $\psi(p, X) = \omega_1$.

Now let us put $S = \{p_\nu: \nu < \omega_1\} \cup \{p\}$ and let G be any neighbourhood of p in S. Then there is a $\nu < \omega_1$ such that $G \supset U_\nu \cap S$. But obviously $U_\nu \cap S = \{p_\mu: \nu < \mu < \omega_1\}$, hence $|S \setminus G| \leqslant |S \setminus U_\nu| \leqslant \omega$, i.e. any open cover of S contains a member, whose complement is countable; consequently S must be Lindelöf.

PROOF OF THEOREM 1. Now let X be an uncountable compact space, and let $A \subset X$, $|A| = \omega_1$. If $|\overline{A}| = \omega_1$, too, we are done. Hence we assume $|\overline{A}| > \omega_1$.

If $p \in \overline{A}$ and $\chi(p, \overline{A}) = \omega$, then we can choose a sequence from A converging to p, hence

$$\left|\left\{p \in \overline{A}: \chi(p, \overline{A}) \leqslant \omega\right\}\right| \leqslant \omega_1^\omega = \omega_1 \quad \text{(by CH)}.$$

Applying the lemma, we see that if $\chi(p, \overline{A}) = \omega_1$, for some $p \in \overline{A}$, then we are again done, hence we might assume $\chi(p, \overline{A}) \neq \omega_1$ for all $p \in \overline{A}$. Consequently if $p \in \overline{A}$ is such that it has a neighbourhood U (in $\overline{A}$) with $|U| \leqslant \omega_1$, then we have $\psi(p, \overline{A}) = \chi(p, \overline{A}) \leqslant \omega_1$, hence $\chi(p, \overline{A}) \leqslant \omega$.

Now let G be the union of all open subsets of $\overline{A}$ of cardinality $\leqslant \omega_1$. Then from what we proved above, for every point p of G we have $\chi(p, \overline{A}) \leqslant \omega$, hence $|G| \leqslant \omega_1$. By the definition of G, $\overline{A} \setminus G$ is compact and has no isolated points (i.e. $\chi(p, \bar{A} \setminus G) \geqslant \omega$ for all $p \in \bar{A} \setminus G$). By Proposition 2, there is a set $C \subset \bar{A} \setminus G$ with $|C| = \omega$ and $|\bar{C}| \geqslant \omega_1$. Again, we might assume that $|\bar{C}| > \omega_1$.

Since $\bar{C}$ has a countable dense subset, we have $w(\bar{C}) \leqslant 2^{\omega} = \omega_1$; consequently $\chi(p, \bar{C}) \leqslant \omega_1$, for all $p \in \bar{C}$.

Moreover in the same way as above it follows that

$$\left|\left\{p \in \bar{C}: \chi\left(p, \bar{C}\right) \leqslant \omega\right\}\right| \leqslant \omega_1.$$

Hence there is a $p \in \bar{C}$ with $\chi(p, \bar{C}) = \omega_1$, and thus, by the lemma, there is a Lindelöf subspace of cardinality ω_1.

REMARK. A similar argument can be used to show that, assuming GCH, for any κ and any compact space of cardinality $> \kappa$ there is a κ-Lindelöf subspace of cardinality κ^+. It should be interesting to decide whether the first κ in the conclusion could be lowered (perhaps even to ω).

It is natural to raise the question why we have not asked about the existence of compact (i.e. closed) subspaces of cardinality ω_1 of uncountable compact spaces. The space βN, however, shows that the answer to this question is negative as it is well known that any infinite closed subspace of βN has the cardinality $2^{2^{\omega}}$.

Our next result shows that compact spaces without small infinite closed subsets must be large; under CH they have to have the same cardinality as βN.

THEOREM 2. *Suppose X is compact and every infinite closed subset of X is uncountable. Then $|X| \geqslant 2^{\omega_1}$.*

PROOF. Let C be the set of all nonisolated points of X. First we claim that no point of C is isolated in C, either. Indeed, let $p \in C$ and assume, on the contrary, that U is a closed neighbourhood of p in X such that $U \cap C = \{p\}$. However then U is infinite, because p is not isolated in X, and for any countably infinite $A \subset U$ the set $A \cup \{p\}$ is closed because its complement in U consists of isolated points only. Thus C must be dense in itself.

However, for no point $p \in C$ can we have $\chi(p, C) = \omega$, because in this case we could select from $C \setminus \{p\}$ an ordinary sequence $\langle p_n: n \in \omega\rangle$ of different points converging to p, and then $\{p_n: n \in \omega\} \cup \{p\}$ would be a countably infinite closed subset of C, and thus of X. But then for each $p \in C$ we have $\chi(p, C) \geqslant \omega_1$, consequently, by the Čech-Pospišil theorem, $|X| \geqslant |C| \geqslant 2^{\omega_1}$.

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MTA MATEMATIKAI KUTATÓ INTÉZETE, H-1053 BUDAPEST, REÁLTANODA U. 13-15, BUDAPEST, HUNGARY

EÖTOVÖS LORÁND TUDOMÁNYEGYETEM, H-1088 BUDAPEST, MÚSEUM KRT. 6-8, BUDAPEST, HUNGARY