ON MAXIMALITY OF GORENSTEIN SEQUENCES

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Abstract. It is well known that if $A$ is a Gorenstein ring, then every ideal generated by a regular sequence $x \subseteq A$ has irreducible (minimal) primary components. This feature led us to define a Gorenstein sequence of a ring $A$ to be any ordered regular sequence $x = \{x_1, \ldots, x_r\} \subseteq A$ such that for every $i \in \{1, \ldots, r\}$ the ideal $(x_1, \ldots, x_i)$ has irreducible minimal primary components. We showed for Gorenstein sequences (G-sequences for short) some parallels of well-known properties of regular sequences and moreover by means of G-sequences we gave the following natural characterization of local Gorenstein rings: “A local ring $(A, m)$ is Gorenstein iff $m$ contains a G-sequence of length $= K - \dim A$”.

In this note we are going to give some information about “maximality” of G-sequences in a local ring $A$, producing sufficient conditions on $A$ in order that the maximal G-sequences of $A$ all have the same length, i.e. in order to give a “good” definition of G-depth $A$. Furthermore, we will state some results about the G-depth behavior with respect to local flat ring homomorphisms.

0. Notations and preliminary results. Throughout this paper $A$ will denote a commutative noetherian ring with unity. In addition, we will say that the Gorenstein locus of $A$ is open if there exists an ideal $\mathfrak{N} \subseteq A$ such that for every $p \in \text{Spec } A$, $A_p$ is a local Gorenstein ring iff $p \not\subseteq \mathfrak{N}$. In [H. K., Vortrag 6] it has been shown that the Gorenstein locus of a local Cohen-Macaulay (C. M. for short) ring $A$ is open if there exists the canonical module $K_A$ (for the definition and properties of $K_A$ see [H. K., Vortrag 5] where it has also been given a sufficient criterion for the existence of $K_A$). If $K_A$ exists then the ideal $\mathfrak{N} \subseteq A$ which defines the non-Gorenstein locus of $A$ admits an explicit description (see [H. K., Bemerkung and Lemma 6.19]) as the ideal generated by the elements $q(x)$ with $x \in K_A$, $q \in \text{Hom}_A(K_A, A)$, i.e. $\mathfrak{N}$ is the trace ideal of $K_A$ (cf. [De M. I., Chapter 1, §1B]).

In this connection we have the following handy characterization:

0.1. If $A$ is a ring with open Gorenstein locus and $\mathfrak{N} \subseteq A$ is the ideal defining the non-Gorenstein locus of $A$, then an ordered regular sequence $\{x_1, \ldots, x_r\} \subseteq A$ is a G-sequence iff for every $i \in \{1, \ldots, r\}$ and $p \in \text{Ass}(A/(x_1, \ldots, x_i))$ we have $p \not\subseteq \mathfrak{N}$.

Now we recall explicitly two concepts which will be frequently used in the following.
0.2. The G-depth of a local ring \((A, m)\) is the largest number of elements in \(m\) making up a G-sequence of \(A\). If \(m\) does not contain any G-sequence of positive length, we define the G-depth \(A\) to be 0 if the zero ideal of \(A\) has irreducible minimal primary components (i.e. the empty set \(\emptyset \subset A\) makes up a G-sequence of length 0), otherwise G-depth \(A = -\infty\) (cf. \([M_1, \text{Definition 4.1}]\)).

0.3. A ring \(A\) is said to be \(G_n\) if G-depth \(A_p > \min(n, \text{ht } p)\) for every \(p \in \text{Spec } A\) (cf. \([M_1, \text{Definition 4.5}]\)).

In \([M_2, \text{Theorem 2.1}]\) we gave several equivalent characterizations for the \(G_n\) condition, showing in particular that a ring \(A\) is \(G_n\) iff it is \(S_n\) (i.e. depth \(A_p > \min(n, \text{ht } p)\) for every \(p \in \text{Spec } A\)) and \(A_p\) is a local Gorenstein ring for every \(p \in \text{Spec } A\) such that \(\text{ht } p < n\). Here we can add the following:

0.4. If \(A\) is an \(S_n\) ring with open Gorenstein locus and \(\mathfrak{g} \subset A\) is the ideal defining the non-Gorenstein locus of \(A\), then \(A\) is \(G_n\) iff \(\text{ht } \mathfrak{g} > n\).

1. Principal results and examples.

**Proposition 1.1.** Let \(A\) be any local C. M. ring with open Gorenstein locus and let \(\mathfrak{g} \subset A\) be the ideal which defines the non-Gorenstein locus of \(A\); then:

(a) if \(\text{ht } \mathfrak{g} = 0\), \(A\) does not contain any G-sequence;

(b) if \(\text{ht } \mathfrak{g} > 0\), all maximal G-sequences of \(A\) have the same length (exactly equal to \(K - \dim A\) or \(K - \dim A - 1\), according as \(A\) is Gorenstein or not).

**Proof.** (a) \(A\) is not a \(G_0\) ring (cf. 0.4) so it is clear that \(A\) does not contain any G-sequence of length \(= 0\) (cf. \([M_2, \text{Osservazione (iii)}]\)). On the other hand one can easily see that \(A\) does not even contain G-sequences of positive length since if there existed \(x \subset A\) a G-sequence of length \(= r > 0\), then both \(A/(x)\) and \(A\) would be \(G_0\) rings (respectively by \([M_2, \text{Osservazione (iv)}]\) and \([M_3, \text{Lemma 2.2}]\)) and this would clearly give a contradiction.

(b) First of all we observe that if \(A\) is a Gorenstein ring, then manifestly all maximal G-sequences of \(A\) must have the same length exactly equal to \(K - \dim A\) since it follows directly from the definitions that all regular sequences of any (not necessarily local) Gorenstein ring are also G-sequences (cf. \([B, \text{Fundamental Theorem}]\), and \([M_1, \text{Definition 2.1}]\)). Therefore to complete our proof we have only to examine the case \(\mathfrak{g} \notsubset A\). Here \(\text{ht } \mathfrak{g} > 0\) implies that for every \(p \in \text{Spec } A\) such that \(\text{ht } p = 0\), \(A_p\) is a local Gorenstein ring (cf. 0.4): hence all (minimal) primary components of the zero ideal in \(A\) are irreducible, i.e. the empty set \(\emptyset \subset A\) is a G-sequence of length \(= 0\) (cf. \([M_2, \text{Osservazione (vi)}]\)). This just proves our theorem in case \(K - \dim A = 1\) where \(\emptyset\) is actually the only (maximal) G-sequence of \(A\) (we are assuming \(A\) is not Gorenstein), so from now on we can suppose \(K - \dim A > 1\). Since \(A\) is a C. M. ring \(\text{ht } \mathfrak{g} > 0\) implies also that there exists some element \(i \in \mathfrak{g}\) which is regular in \(A\) and so can be completed to a maximal regular sequence of \(A\), say \(x = \{i, x_2, \ldots, x_m\}\) with \(m = K - \dim A\). We want to show that \(x' = \{x_2, \ldots, x_m\}\) is a G-sequence of \(A\); in this connection to say that \(x \subset A\) is a regular sequence means in particular that \(i \not\in \mathfrak{y}'\) for any \(\mathfrak{y}' \in \text{Ass } (A/(x'))\), i.e. \(A_{\mathfrak{y}'}\) is a local Gorenstein ring for every \(\mathfrak{y}' \in \text{Ass } (A/(x'))\) and this means that \(x'\) is a G-sequence, so \(\emptyset\) is not a maximal G-sequence in \(A\). Then let \(y \subset A\) be any maximal G-sequence of positive length \(= s < m\).
(recall that we are assuming $A$ is not Gorenstein): for every $\mathcal{Q} \in \text{Ass} \left( A/(y) \right)$ clearly $\mathcal{Q} \supseteq \mathfrak{g}$ so there exists some element $j \in \mathfrak{g}$ which is a non-zero-divisor modulo $(y)$ such that $(y,j)$ is a regular sequence of length $= s + 1$. If $s < m - 1$, $(y,j)$ can be completed to a maximal regular sequence of $A$, say $(y,j,y_{s+2},\ldots,y_m)$; here applying the same argument as above we can see that $(y,y_{s+2},\ldots,y_m)$ is a $G$-sequence of $A$ containing $y$, contradicting the maximality of $y$, so all maximal $G$-sequences of $A$ actually have length $= K - \dim A - 1$.

**Corollary 1.2.** If $A$ is a local C. M. ring with open Gorenstein locus, then $G$-depth $A$ is the length of any maximal $G$-sequence in $A$ (as usual $G$-depth $A = -\infty$ if $A$ does not contain $G$-sequences of any length).

**Corollary 1.3.** Let $A$ be any $S_n$ local ring with $K - \dim A > n > 1$. If the Gorenstein locus of $A$ is open, then either $A$ does not contain any $G$-sequence or all maximal $G$-sequences of $A$ have length $\geq n - 1$.

**Proof.** Let $\mathfrak{g}$ be the ideal which defines the non-Gorenstein locus of $A$; we can observe that, as in Proposition 1.1, our conclusion and proof depend on height $\mathfrak{g}$. Precisely: $\text{ht } \mathfrak{g} = 0$ implies both $A$ does not contain any $G$-sequence of length $= 0$ (in fact in this case $A$ is not $G_0$) and $A$ does not contain any $G$-sequence of positive length (namely if $x = (x_1,\ldots,x_r)$ would be a $G$-sequence of length $= r > 0$, then for all $i \in \{1,\ldots,r\}$ $x_i$ would be a $G$-sequence which generates an unmixed ideal (cf. [S, Theorem 2.2]), so $A/(x_i)$ and then $A$ (by [R. F., Proposition 3]) would be $S_1$ and $G_0$ rings contradicting the fact that $A$ is not $G_0$).

$\text{ht } \mathfrak{g} > 0$ implies $\emptyset$ is not a maximal $G$-sequence (we can use an argument like that of Proposition 1.1). Then let $x \subseteq A$ be any maximal $G$-sequence of length $= s > 0$, if $s < n - 1$ for every $\mathfrak{b} \in \text{Ass} \left( A/(x) \right)$, $\mathfrak{b} \supseteq \mathfrak{g}$, $(x)$ is an unmixed ideal by [S, Theorem 2.2], so there exists some element $j \in \mathfrak{g}$ which is a non-zero-divisor modulo $(x)$. Considering the regular sequence $(x,j)$, we can conclude, as in Proposition 1.1:

**Remark I.** From [M3, Lemma 2.2] and [R. F., Proposition 3], we can deduce some information about the existence of $G$-sequences in a local ring $A$ without having resort to the hypothesis that the Gorenstein locus of $A$ is open; precisely we can say:

(i) In any local non-$G_0$ ring $A$ which satisfies the "saturated chain condition on prime ideals", there exist no $G$-sequences.

(ii) In any local non-$G_0$ ring $A$ there exist no $G$-sequences (of any length) which generated unmixed ideals, but a priori we do not have any information about possible $G$-sequences which generated mixed ideals.

**Remark II.** From the proof of Corollary 1.3 we can deduce that in a local $S_n$ ring $A$ ($K - \dim A > n > 1$) which is $G_0$ and has open Gorenstein locus, there exist $G$-sequences of length $= \text{depth } A - 1$ but a priori we cannot say if this must be the length of every maximal $G$-sequence of $A$, so actually we do not know whether there may exist maximal $G$-sequences of different lengths in $A$.

Recall explicitly the following notation introduced in [W.I.T.O., Definition 1.7].
Definition 1.4. A ring homomorphism \( \varphi: A \to B \) is Gorenstein if it is flat and has Gorenstein fibres.

Lemma 1.5. Let \( A \) be any ring with open Gorenstein locus. Then, for every Gorenstein homomorphism \( \varphi: A \to B \), the Gorenstein locus of \( B \) is open.

Proof. Let \( f: Y = \text{Spec } B \to \text{Spec } A = X \) be the induced morphism and let \( U \subseteq X \) be the Gorenstein locus of \( A \). For every \( \mathfrak{B} \in f^{-1}(U) \), \( B_{\mathfrak{B}} \) is Gorenstein; namely, putting \( \nu = f(\mathfrak{B}) \), clearly \( \nu \subseteq U \) so the local homomorphism \( \psi: A_{\mathfrak{B}} \to B_{\nu} \) is flat with \( A_{\mathfrak{B}} \) Gorenstein (since \( \nu \subseteq U \)) and \( B_{\mathfrak{B}}/\nu B_{\mathfrak{B}} \) Gorenstein (since it is a localization of the fibre of \( \varphi \) at \( \nu \)). On the other hand, for every \( \mathfrak{Q} \in Y - f^{-1}(U) \), \( B_{\mathfrak{Q}} \) is not Gorenstein since putting \( q = \mathfrak{Q} \cap \mathfrak{A} \) clearly \( q \not\subseteq U \), so \( A_{q} \) is not Gorenstein. Therefore the Gorenstein locus of \( B \) is precisely \( f^{-1}(U) \) which, by the hypothesis, is clearly open.

Proposition 1.6. Let \((A, \mathfrak{m})\) be any local C. M. ring with open Gorenstein locus. Then for every local Gorenstein homomorphism \( \varphi: A \to B \), we have

\[
\text{G-depth } B = \text{G-depth } A + \text{G-depth } B/\mathfrak{m} B.
\]

Proof. Observe that under the given hypotheses not only \( B \) is a (local) C. M. ring (cf. [D, Corollary 5.1]) but also its Gorenstein locus is open (cf. Lemma 1.5): therefore \( \text{G-depth } B \) is actually well defined (cf. Corollary 1.2). In addition, we notice that if \( A \) is Gorenstein, then there is nothing to prove, since in that case \( B \) is a (local) Gorenstein ring by [W.I.T.O., Theorem 1], so clearly

\[
\text{G-depth } B = K \cdot \text{dim } B = K \cdot \text{dim } A + K \cdot \text{dim } B/\mathfrak{m} B
\]

Then, to show our contention, we only have to examine the case \( A \) is not Gorenstein which, according to Proposition 1.1, splits into \( \text{G-depth } A = -\infty \) and \( \text{G-depth } A = K \cdot \text{dim } A - 1 \) (automatically \( \geq 0 \)). If \( \text{G-depth } A = -\infty \), then we cannot have \( \text{G-depth } B \geq 0 \), since that would mean \( B \) is at least \( \mathcal{G}_0 \) and hence also \( A \) would be at least \( \mathcal{G}_0 \) (cf. [M1, Theorem 5.1]), contradicting \( \text{G-depth } A = -\infty \); then

\[
\text{G-depth } B = -\infty = -\infty + K \cdot \text{dim } B/\mathfrak{m} B
\]

If \( 0 \leq \text{G-depth } A = K \cdot \text{dim } A - 1 \), then \( B \) is a \( \mathcal{G}_0 \) ring (cf. [M1, Corollario 5.2]), that is \( \text{G-depth } B \geq 0 \). Moreover, since we are assuming \( A \) is not Gorenstein, we cannot have \( \text{G-depth } B = K \cdot \text{dim } B \) (cf. [W.I.T.O., Theorem 1]); thus again

\[
\text{G-depth } B = K \cdot \text{dim } B - 1 = K \cdot \text{dim } A - 1 + K \cdot \text{dim } B/\mathfrak{m} B
\]

Remark III. Observe that if the local C. M. ring \((A, \mathfrak{m})\) has the canonical module \( K_A \), then every local flat ring homomorphism \( \varphi: A \to B \) such that
$B/mB$ is Gorenstein (i.e. $K_B \simeq K_A \otimes_A B$ (cf. [H.K., Satz 6.14.])) is actually a Gorenstein homomorphism. Namely, using the same notations as in Lemma 1.5, for any $p \in X$, the fibre of $\varphi$ at $p$ is Gorenstein since (cf. [E.G.A., IV$_2$, Lemma 7.3.2]) for every $\mathfrak{p} \in f ^{-1}(p)$, $B_{\mathfrak{p}}/pB_{\mathfrak{p}}$ is Gorenstein, being

$$(K_{A_p}) \otimes_{A_p} B_{\mathfrak{p}} = (K_A) _X \otimes_A A_{\mathfrak{p}} \otimes_{A_p} B_{\mathfrak{p}} = K_A \otimes_A A_{\mathfrak{p}} \otimes_{A_p} B_{\mathfrak{p}} = K_A \otimes_A B_{\mathfrak{p}}$$

(cf. [R, (3) Theorem] and [H.K., Korollar 5.25]). Notice that the hypothesis “$B/mB$ is Gorenstein” cannot be avoided; namely, we can easily see that Proposition 1.6 does not hold for a local flat homomorphism $\varphi: A \to B$ of complete, equidimensional local rings such that $A$ is Gorenstein and $B$ is $G_0$ but not Gorenstein (e.g., fix a field $k$, take

$$A = k[[UN]] \quad (N = \text{l.c.m. } (6, 7, 8, 9)),$$

$$B = k[[x,y,z,t]]/(t^2 - x^2 - yt, y^2 - xz, yz - xt),$$

and $\varphi$ the inclusion map (which is clearly flat and local); what we get is $G$ - depth $A = 1$, $G$ - depth $B = 0$ ($B$ is a 1-dimensional complete integral domain which is not Gorenstein (cf. [K, Theorem]), and $G$ - depth $B/mB = -\infty$).

Finally we are going to list explicitly some examples of local flat ring homomorphisms for which Proposition 1.6 holds.

1.7. Let $(A, m)$ be any local C.M. ring with residue field $k$ such that $K_A$ exists. Then:

(i) If $x$ is an indeterminate, for every maximal ideal $\mathfrak{m} \subset A[x]$ such that $\mathfrak{m} \cap A = m$, we have $G$ - depth $A[x]_{\mathfrak{m}} = G$ - depth $A + 1$ (in fact the fibre $k \otimes_A A[x]_{\mathfrak{m}}$ is isomorphic to the 1-dimensional Gorenstein ring $k[x]$ localized at the maximal ideal $\mathfrak{m} = A[x]_k[x]$ (cf. [G.S., Example 12.1])).

(ii) If $F$ is a finite abelian group, for every maximal ideal $\mathfrak{m} \subset A[F]$, we have $G$ - depth $A[F]_{\mathfrak{m}} = G$ - depth $A$ (in fact $k \otimes_A A[F]_{\mathfrak{m}}$ is isomorphic to the 0-dimensional Gorenstein ring $k[F]$ localized at the maximal ideal $\mathfrak{m} = A[F]_k[F]$ (cf. [P, Corollaire 2])).

(iii) If $b^A$ is the henselization of $A$ with respect to $m$, we have $G$ - depth $b^A = G$ - depth $A$ (in fact $k \otimes_A b^A = k \otimes_A b^A / m b^A = k$  (cf. [E.G.A., IV$_4$, Theorem 18.6.6])).

(iv) If $\hat{A}$ is the $m$-adic completion of $A$, we have $G$ - depth $\hat{A} = G$ - depth $A$ (in fact $k \otimes_A \hat{A} \simeq \hat{A}/m \hat{A} \simeq k$ (cf. [D, §6.A5])).

(v) If $A[[x]]$ is the formal power series ring (in one indeterminate) over $A$, we have $G$ - depth $A[[x]] = G$ - depth $A + 1$ (in fact the fibres of $A \to A[[x]]$ are canonically isomorphic to the fibres of $B \to B[[x]]$ where $B$ is a local Gorenstein ring such that $A = B/b$ (cf. [R, (3) Theorem]) at the prime ideals of $B$ containing $b$, and $B \to B[[x]]$ is a local flat ring homomorphism whose fibre at the closed point of Spec $B$ is equal to $k \otimes_B B[[x]]$ which is clearly a (local) 1-dimensional Gorenstein ring (cf. [G.S., Theorem 9.8] and [W.I.T.O., Theorem 2])).
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