A NOTE ON WALSH-FOURIER SERIES

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Abstract. It is shown that the double sequence \( \{ \lambda_{mn} \} \) with \( \lambda_{mn} = 1 \) if \( n < m \) and 0 otherwise is an \( L^p \) multiplier for the Walsh system in two dimensions only if \( p = 2 \). This result is then used to show that the one-dimensional trigonometric system and the Walsh system are nonequivalent bases of the Banach space \( L^p[0, 1] \), and hence have different \( L^p \) multipliers, \( 1 < p < \infty, p \neq 2 \).

1. Let \( \{ \lambda_{mn} \}_{-\infty < m, n < \infty} \) be the double sequence defined by \( \lambda_{mn} = 1 \) if \( n < m \) and 0 otherwise. For

\[
\sum_{m,n=-\infty}^{\infty} a_{mn} e^{2\pi i (mx + ny)} \in L^p \left([0, 1] \times [0, 1]\right),
\]

let

\[
T_1 f \sim \sum_{m,n=-\infty}^{\infty} a_{mn} \lambda_{mn} e^{2\pi i (mx + ny)}.
\]

It is well known that \( T_1 \) is bounded on \( L^p([0, 1] \times [0, 1]), 1 < p < \infty \). This is a consequence of the one-dimensional result of M. Riesz for the conjugate function \([6, 1, p. 253]\). The \( L^p \) boundedness of \( T_1 \) was used, for example, in C. Fefferman's proof of the almost everywhere convergence of double Fourier series [1].

We now turn our attention to the Walsh system \( \{ w_n \} \). For

\[
\sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y) \in L^p \left([0, 1] \times [0, 1]\right),
\]

consider the corresponding operator \( T_2 \) defined by

\[
T_2 f \sim \sum_{m,n=0}^{\infty} a_{mn} \lambda_{mn} w_m(x) w_n(y).
\]

Because of the great similarity between the Walsh system and the trigonometric system, one would expect \( T_2 \) to be bounded on \( L^p([0, 1] \times [0, 1]), 1 < p < \infty \). However, this is not the case. In \( \S 2 \) we will show that \( T_2 \) is not bounded on \( L^p \) except for \( p = 2 \). This result is then used in \( \S 3 \) to give a negative answer to a question of P. Enflo: Are the trigonometric system and the Walsh system equivalent bases of the Banach space \( L^p[0, 1], 1 < p < \infty, p \neq 2 \)? Finally in \( \S 4 \) we will deduce from the nonequivalence that the
one-dimensional Walsh system and trigonometric system have different $L^p$ multipliers, $1 < p < \infty$, $p \neq 2$.

2. Let $\{r_n\}_{n \geq 0}$ denote the Rademacher functions and $\{w_n\}_{n \geq 0}$ the Walsh functions defined on $I = [0, 1]$. For any two real numbers with dyadic expansions $a = \sum_{j=-\infty}^{\infty} \alpha_j 2^j, \quad b = \sum_{j=-\infty}^{\infty} \beta_j 2^j, \quad \alpha_j, \beta_j = 0$ or $1$, let $a + b = \sum_{j=-\infty}^{\infty} [\alpha_j - \beta_j] 2^j$. It is understood that we use the finite representation in the case of a dyadic rational. Basic properties of the Walsh functions can be found in [2].

**Theorem 1.** $T_2$ is bounded on $L^p(I^2)$ if and only if $p = 2$.

**Proof.** The case $p = 2$ is trivial by Parseval's formula. It is sufficient to show that $T_2$ is not bounded on $L^p$ for $p < 2$, for then the theorem will follow by a duality argument.

Instead of dealing with $T_2$ we consider an equivalent operator $T'_2$ defined on $L^p(I^2)$ as follows. Let $A = \{(m, n) : 0 < n < m, m = 0, 1, \ldots \}$. For $f = \sum_{m,n} a_{mn} w_m(x) w_n(y)$, let $T'_2 f = \sum_{(m,n) \in A} a_{mn} w_m(x) w_n(y)$. Suppose $g$ is the function on $I^2$ defined by $g(x, y) = f(x, x + y)$. Since $w_n(x + y) = w_n(x) w_n(y)$ and $w_{m+n} = w_m w_n$, we have $T'_2 g(x, y) = T_2 f(x, x + y)$. Therefore the boundedness of $T_2$ is equivalent to that of $T'_2$.

We will define a sequence of functions $\{f_k\}$ on $I^2$ and show that

$$\|T'_2 f_k\|_p / \|f_k\|_p \to 0 \quad \text{as} \quad k \to \infty.$$  

Let

$$f_k = \sum_{l=1}^{k} f_{kl}$$

where

$$f_{kl}(x, y) = 2^{-k} r_{l-1}(x) \prod_{j=0}^{k-1} (1 + r_j(y)).$$

We first note that

$$\|f_{kl}\|_p^p = \left| \int 2^{-k} \prod_{j=0}^{k-1} (1 + r_j(y)) \, dy \right|^p = 2^{-kp} \int_{[0,2^{-k}]} 2^{kp} \, dy = 2^{-k}.$$  

Moreover, since $\int [\sum_{l=0}^{k} r_l(x)]^p \, dx \leq C_p k^{p/2}$ by Khintchin's inequality [6, I, p. 213], we have

$$\|f_k\|_p^p = \left| \int \left[ \sum_{l=0}^{k} r_l(x) \right]^p \, dx \right|^p \leq C_p k^{p/2} 2^{-k},$$  

where $C_p$ denotes a constant depending only on $p$.

On the other hand, for $1 < l < k$,

$$T'_2 f_{kl}(x, y) = 2^{-k} \sum_{n < m+n, m} r_{l-1}(s) w_m(x) \int r_{l-1}(t) w_n(t) \, dt \cdot \prod_{j=0}^{k-1} (1 + r_j(t)) w_n(x) w_n(y).$$
Now
\[ \int r_{l-1}(s)w_m(s) \, dx = 1 \text{ if } m = 2^{l-1} \]
on otherwise. Also
\[ \int \prod_{j=0}^{k-1} (1 + r_j(t))w_n(t) \, dt = 1 \text{ if } n < 2^k \]
\[ = 0 \text{ if } n \geq 2^k. \]
Therefore
\[ (3) \quad T_{2^k}f_{kl}(x, y) = 2^{-k} \sum_{n < 2^{l-1} + n; n < 2^k} w_n(y)w_{2^{l-1}}(x). \]
Let \( n = \sum_{j=0}^{\infty} \varepsilon_j 2^j \) with \( \varepsilon_j = 0 \) or 1. We observe that \( n < 2^{l-1} + n \) if and only if \( \varepsilon_{l-1} = 0 \). Therefore
\[ \sum_{n < 2^{l-1} + n; n < 2^k} w_n(y) = \prod_{0 < j < l; j \neq l-1} (1 + r_j(y)) \prod_{0 < j < k} (1 + r_j(y + 2^{-j})) \]
\[ = \frac{1}{2} \left[ \prod_{0 < j < k} (1 + r_j(y)) + \prod_{0 < j < k} (1 + r_j(y + 2^{-j})) \right]. \]
From (3) and (4), we have
\[ T_{2^k}f_{kl}(x, y) = \frac{1}{2} \left[ f_{kl}(x, y) + f_{kl}(x, y + 2^{-l}) \right], \]
and hence
\[ T_{2^k}f_k(x, y) = \frac{1}{2} f_k(x, y) + \frac{1}{2} \sum_{l=1}^{k} f_{kl}(x, y + 2^{-l}). \]
Since \( f_k(x, y) \) and \( f_{kl}(x, y + 2^{-l}), \ l = 1, \ldots, k, \) have mutually disjoint supports, it follows from (1) that
\[ (5) \quad \|T_{2^k}f_k\|_p > 2^{-p} \sum_{l=1}^{k} \|f_{kl}\|_p = 2^{-p} k 2^{-k}. \]
Combining (2) and (5), we obtain, for \( p < 2, \)
\[ \|T_{2^k}f_k\|_p/\|f_k\|_p > 2^{-1} C_p^{-1} k^{(1/p - 1/2)} \to \infty \text{ as } k \to \infty. \]
This completes the proof of Theorem 1.

3. It is known that \( \{ \cos \pi nx \} \) and \( \{ w_n \} \) are bases of the Banach space \( L^p(I) \), \( 1 < p < \infty. \) (See [6, I, p. 266] and [4].) We say that the sequences \( \{ u_n \}, \{ v_n \} \) of a Banach space are equivalent if for every sequence of numbers \( \{ a_n \}, \sum_{n=0}^{\infty} a_n u_n \) converges if and only if \( \sum_{n=0}^{\infty} a_n v_n \) converges. R. Askey, S. Wainger and J. E. Gilbert showed that \( \{ \cos \pi nx \} \) and certain classical orthonormal sequences are equivalent in \( L^p(I) \), \( 1 < p < \infty. \) (See [3].) We have the following

**Theorem 2.** Let \( 1 < p < \infty. \) \( \{ \cos \pi nx \} \) and \( \{ w_n \} \) are equivalent bases of \( L^p(I) \) if and only if \( p = 2. \)

**Proof.** Again the case \( p = 2 \) is trivial by Parseval’s formula. Suppose they
were equivalent in $L^p(I)$, $p \neq 2$. From this it would follow that $\{e^{2\pi i n x}\}_{n \geq 0}$ and $\{w_n\}$ are also equivalent in $L^p(I)$. (See [6, I, p. 253].) By the Banach-Steinhaus theorem, there exist constants $C_p$, $C'_p > 0$ such that for any sequence of numbers $\{a_n\}$,

$$C_p^{-1} \left\| \sum_{n=0}^{N} a_n e^{2\pi i n x} \right\|_p \leq \left\| \sum_{n=0}^{N} a_n w_n \right\|_p \leq C'_p \left\| \sum_{n=0}^{N} a_n e^{2\pi i n x} \right\|_p, \quad N > 0.$$  

(See [5, p. 70].) Let $\{a_{mn}\}$ be any double sequence of numbers. Applying (6) first to the $x$-variable and then to the $y$-variable, we obtain

$$C_p^{-2} \left\| \sum_{m,n=0}^{N} a_{mn} e^{2\pi i (m+n y)} \right\|_p \leq \left\| \sum_{m,n=0}^{N} a_{mn} w_m(x) w_n(y) \right\|_p \leq C'_p \left\| \sum_{m,n=0}^{N} a_{mn} e^{2\pi i (m+n y)} \right\|_p, \quad N > 0.$$  

(7)

Now, it follows from the corresponding one-dimensional result that for any function in $L^p(I^2)$, the square partial sums of both its trigonometric Fourier series and Walsh-Fourier series converge in $L^p(I^2)$. Therefore (7) implies the following: for any sequence of numbers $\{a_{mn}\}$, $f \sim \sum_{m,n=0}^{\infty} a_{mn} e^{2\pi i (m+n y)}$ for some $f \in L^p(I^2)$ if and only if $g \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y)$ for some $g \in L^p(I^2)$.

Suppose $g \in L^p(I^2)$ with $g \sim \sum_{m,n=0}^{\infty} a_{mn} w_m(x) w_n(y)$. Then $f \sim \sum_{m,n=0}^{\infty} a_{mn} e^{2\pi i (m+n y)} \in L^p(I^2)$. Hence

$$T_1 f \sim \sum_{m,n=0}^{\infty} a_{mn} \lambda_{mn} e^{2\pi i (m+n y)} \in L^p(I^2),$$

which implies

$$T_2 g \sim \sum_{m,n=0}^{\infty} a_{mn} \lambda_{mn} w_m(x) w_n(y) \in L^p(I^2).$$

Therefore $T_2$ maps $L^p(I^2)$ into $L^p(I^2)$. By the closed graph theorem, $T_2$ is bounded on $L^p(I^2)$, contradicting Theorem 1. This proves Theorem 2.

4. Let $\{u_n\}$ be one of the sequences $\{w_n\}$, $\{\cos \pi n x\}$ or $\{e^{2\pi i n x}\}_{n \geq 0}$. $M(L^p, \{u_n\})$ denotes the collection of all sequences $\{\lambda_n\}$ such that $f \sim \sum_{n=0}^{\infty} a_n u_n \in L^p(I)$ implies $g \sim \sum_{n=0}^{\infty} \lambda_n a_n u_n \in L^p(I)$. We will deduce from Theorem 2 the following

**Theorem 3.** $M(L^p, \{\cos \pi n x\}) \neq M(L^p, \{w_n\})$, $1 < p < \infty$, $p \neq 2$.

We note that in general two nonequivalent bases of $L^p(I)$ may have the same multipliers. See, for example, [5, p. 484 and p. 546].

**Proof.** Suppose they were equal. Since

$$M(L^p, \{\cos \pi n x\}) = M(L^p, \{e^{2\pi i n x}\}_{n \geq 0}),$$

we have $M(L^p, \{w_n\}) = M(L^p, \{e^{2\pi i n x}\}_{n \geq 0})$. Let $\sum_{n=0}^{\infty} a_n e^{2\pi i n x} \in L^p$. For every $t \in [0, 1]$,

$$\left\| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \right\|_{L^p} = \left\| \sum_{n=0}^{\infty} a_n e^{2\pi i n x} \right\|_{L^p},$$

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so \(\{e^{2\pi int}\} \in M(L^p, \{e^{2\pi int}\})\), and hence belongs to \(M(L^p, \{w_n\})\). We assert that, moreover, there is a constant \(C_p\), depending only on \(p\), such that for every \(\sum_{n=0}^{\infty} a_n w_n \in L^p\),

\[
(8) \quad \left\langle \int \sum_{n=0}^{\infty} a_n e^{2\pi int} w_n(x) \right\rangle^p \leq C_p \int \left\| \sum_{n=0}^{\infty} a_n w_n(x) \right\|^p dx, \quad t \in [0, 1].
\]

We will prove (8) by contradiction. Suppose there was no such constant. Then there would exist \(\{t_k\} \subset [0, 1]\), \(\{a^{(k)}_{n}\}_{n,k \geq 0}\) and integers \(0 \leq N_0 < N_1 < \ldots\) such that

\[
\left\| \sum_{n=0}^{2^{N_k} - 1} a^{(k)}_n w_n \right\|_p = 1 \quad \text{and} \quad \left\| \sum_{n=0}^{2^{N_k} - 1} a^{(k)}_n e^{2\pi int} w_n \right\|_p > 2^{2k}.
\]

Observe that for \(n = 0, 1, \ldots, 2^{N_k} - 1\), \(n + 2^{N_k} = n + 2^{N_k} \in [2^{N_k}, 2^{N_k} + 1) \subset [2^{N_k}, 2^{N_k} + 1)\). Define a sequence \(\{\lambda_n\}\) by

\[
\lambda_n = \begin{cases} 
0 & \text{if } 0 < n < 2^{N_0}, \\
2^{-k} e^{2\pi int} & \text{if } 2^{N_k} < n < 2^{N_{k+1}} , \quad k = 0, 1, \ldots.
\end{cases}
\]

Then, for \(\sum_{n=0}^{\infty} a_n e^{2\pi int} \in L^p\),

\[
\left\| \sum_{n=0}^{2^{N_k} - 1} \lambda_n a^{(k)}_n e^{2\pi int} \right\|_p \leq \sum_{k=0}^{\infty} 2^{-k} \left\| \sum_{n=2^{N_k}}^{2^{N_k} - 1} a^{(k)}_n e^{2\pi int} \right\|_p
\]

\[
< C_p \left\| \sum_{n=0}^{\infty} a^{(k)}_n e^{2\pi int} \right\|_p.
\]

(See [6, I, p. 266].) Hence \(\{\lambda_n\} \in M(L^p, \{e^{2\pi int}\})\). On the other hand, for \(k = 0, 1, \ldots\),

\[
\left\| \sum_{n=0}^{2^{N_k} - 1} a^{(k)}_n w_{2^{N_k} + n} \right\|_p = 1,
\]

whereas

\[
\left\| \sum_{n=0}^{2^{N_k} - 1} a^{(k)}_n \lambda_{2^{N_k} + n} w_{2^{N_k} + n} \right\|_p = 2^{-k} \left\| \sum_{n=0}^{2^{N_k} - 1} a^{(k)}_n e^{2\pi int} w_n \right\|_p > 2^k.
\]

By the closed graph theorem, \(\{\lambda_n\} \not\in M(L^p, \{w_n\})\), contradicting our assumption. This proves (8). Similarly we can show that there is a constant \(C_p\) such that for every \(\sum_{n=0}^{\infty} a_n e^{2\pi int} \in L^p\),

\[
(9) \quad \int \sum_{n=0}^{\infty} a_n w_n(x) e^{2\pi int} \, dt \leq C_p \int \sum_{n=0}^{\infty} a_n e^{2\pi int} \, dt, \quad x \in [0, 1].
\]

We will now show that (8) and (9) imply the equivalence of \(\{w_n\}\) and \(\{e^{2\pi int}\}\). To see this, let \(a_1, \ldots, a_N\) be any numbers. From (8), we have

\[
\int \int \sum_{n=0}^{N} a_n e^{2\pi int} w_n(x) \, dx dt \leq C_p \int \sum_{n=0}^{N} a_n w_n(x) \, dx.
\]
From (9), we have
\[ \int \left| \sum_{n=0}^{N} a_n e^{2\pi int} \right|^p dt \leq C_p^p \int \left| \sum_{n=0}^{N} a_n w_n(x) e^{2\pi int} \right|^p dt dx. \]

Therefore \( \| \sum_{n=0}^{N} a_n e^{2\pi int} \|_p \leq C_p C_p' \| \sum_{n=0}^{N} a_n w_n \|_p \). Similarly, we have
\[ \left\| \sum_{n=0}^{N} a_n w_n \right\|_p \leq C_p C_p' \left\| \sum_{n=0}^{N} a_n e^{2\pi int} \right\|_p. \]

This shows \( \{w_n\} \) and \( \{e^{2\pi int}\}_{n \geq 0} \) are equivalent, which contradicts Theorem 2. This completes the proof of Theorem 3.

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References