THE NONEXISTENCE OF CERTAIN INVARIANT MEASURES
PAUL ERDÖS AND R. DANIEL MAULDIN

Abstract. It is shown that there does not exist an uncountable group $G$ and a nontrivial, $\sigma$-finite, countably additive measure defined on all subsets of $G$ which is left-invariant.

The purpose of this note is to resolve a point left unclear in a recent paper of F. Terpe [1] and its review [2]. In [1], F. Terpe shows that the existence of a certain "maximal" integral is equivalent to the existence of a nontrivial countably additive $\sigma$-finite measure $m$, defined on all subsets of the interval $I = [0, 1)$ and invariant under translation mod 1. In his review [2] of this paper, J. C. Oxtoby points out that the proof given there for the nonexistence of such a measure tacitly presupposes that the $\sigma$-field $2^I \times 2^I$ of subsets of $I \times I$ generated by generalized rectangles is invariant under the shear map $S$, where $S(x, y) = (x + y, y)$ and addition is mod 1, and that by a theorem of Iwanik [3] this instance of Weil's measurability condition is satisfied if and only if all subsets of $I \times I$ belong to $2^I \times 2^I$. Thus, Terpe's reasoning actually established the nonexistence of $m$ only under the hypothesis $2^I \times I = 2^I \times 2^I$. Finally, Oxtoby points out in his review that $2^I \times I = 2^I \times 2^I$ is implied by CH, but that CH makes the group argument unnecessary. Oxtoby ends his review by stating that the situation is unclear without CH.

We give a short argument below to show that no such hypothesis is needed.

Theorem. Suppose $G$ is an uncountable group and $\mu$ is a $\sigma$-finite countably additive left-invariant measure defined on all subsets of $G$. Then $\mu$ is trivial.

Proof. Let $M$ be a subgroup of $G$ of cardinality $\aleph_1$. Let $R$ be the family of all right cosets of $M$ and let $A$ be a subset of $G$ which intersects each set in $R$ in exactly one point.

Let $\mathcal{K} = \{mA : m \in M\}$. Then $\mathcal{K}$ is a family of $\aleph_1$ disjoint sets covering $G$ and if $H_1$ and $H_2$ belong to $\mathcal{K}$, then $H_2$ is a left translate of $H_1$.

Let $\{K_n\}_{n=1}^\infty$ be a sequence of sets of finite measure covering $G$. For each $n$, the sets of the form $K_n \cap H$, where $H \in \mathcal{K}$ form a decomposition of $K_n$ and therefore there are not uncountably many $H$'s with $\mu(K_n \cap H) > 0$.

Thus, there is a set $H_0$ in $\mathcal{K}$ with $\mu(K_n \cap H_0) = 0$ for each $n$. Therefore, $\mu(H) = 0$ for all $H \in \mathcal{K}$. This implies that $\aleph_1$ is a real-valued measurable...
cardinal. But, assuming the axiom of choice (which we are in this paper), it is known that $\kappa_1$ is not measurable [4].

REFERENCES

2. J. C. Oxtoby, Math Reviews 49 #10861.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32611