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ON A COVERING PROPERTY OF CONVEX SETS

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ABSTRACT. Let $\{K_1, K_2, \ldots\}$ be a class of compact convex subsets of euclidean n-space with the property that the set of their diameters is bounded. It is shown that the sets K_i can be rearranged by the application of rigid motions so as to cover the total space if and only if the sum of the volumes of all the sets K_i is infinite. Also, some statements regarding the densities of such coverings are proved.

If $\{K_i\} = \{K_1, K_2, \ldots\}$ is a class of compact convex sets of euclidean n-dimensional space R^n, we say that $\{K_i\}$ can cover R^n, or that $\{K_i\}$ permits a space covering, if there are rigid motions σ_i so that $R^n \subset \bigcup_{i=1}^{\infty} \sigma_i K_i$. The volume of K_i will be denoted by $v(K_i)$, and the diameter by $d(K_i)$. In a recent report [1], G. D. Chakerian has discussed the problem of finding necessary and sufficient conditions on $v(K_i)$ and $d(K_i)$ in order that a class $\{K_i\}$ permits a space covering. In the joint paper [2] we have proved, using an idea of G. T. Sallee, the special case $n = 2$ of the following theorem:

THEOREM 1. *Let $\{K_i\}$ $(i = 1, 2, \ldots)$ be a class of compact convex subsets of R^n with the property that for some constant M and for $i = 1, 2, \ldots,$*

$$d(K_i) \leqslant M. \tag{1}$$

Then $\{K_i\}$ can cover R^n if and only if

$$\sum_{i=1}^{\infty} v(K_i) = \infty. \tag{2}$$

In the present paper this theorem will be proved in full generality. First we formulate and prove a theorem regarding coverings of the unit cube by a finite number of intervals (Theorem 2). From this theorem it will not only be possible to deduce Theorem 1, but also a more precise statement involving the densities of such coverings (Corollary 1). It should be mentioned that there is a kind of a "dual" of Theorem 2 concerning packings of convex sets rather than coverings (see Kosiński [5]). Theorem 3 shows the possibility of space coverings of density 1 by rather general collections of intervals. As a consequence of this theorem we can deduce another statement (Corollary 2) that establishes, under stronger assumptions than those in Corollary 1, a smaller bound for the densities of coverings of R^n by convex sets.

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The following preliminary remarks are useful in dealing with this class of problems. We define an interval as a subset of R^n of the form $\{(x_1, x_2, \ldots, x_n)\colon a_i \leqslant x_i \leqslant b_i\}$. It can be shown (see Hadwiger [4] or Chakerian [1]) that for every compact convex set K there exists an interval I with

$$v(I) \geqslant (1/n)^n v(K) \tag{3}$$

and so that K contains a congruent copy of I. Consequently, if $\{K_i\}$ is a class of compact convex sets satisfying (1) and (2) the class $\{I_i\}$ of intervals which can be associated in this way has the same property. It follows immediately that it suffices to prove Theorem 1 only for intervals. Note that the necessity of condition (2) is completely trivial. There is no loss in generality by assuming that $M = 1$ in condition (1), since this can always be achieved by an obvious similarity transformation. Furthermore, if W denotes the unit cube

$$W = \left\{(x_1, x_2, \ldots, x_n)\colon -\tfrac{1}{2} \leqslant x_i \leqslant \tfrac{1}{2}\right\}$$

in R^n and if we wish to prove that a class $\{I_i\}$ of intervals satisfying (1) and (2) can cover R^n, it is sufficient to show that W can be covered by finitely many of these intervals. Indeed, if this is possible one can write R^n as a union of unit cubes and cover successively one cube after the other.

These remarks show that Theorem 1 is an immediate consequence of the following theorem.

THEOREM 2. *If* $I_1, I_2, \ldots, I_m$ *are intervals in* R^n *with* $d(I_i) \leqslant 1$, *and if*

$$\sum_{i=1}^{m} v(I_i) \geqslant 2 \cdot 3^{n-1} - 1, \tag{4}$$

then W can be covered by these intervals using only rigid motions that transform intervals into intervals.

PROOF. For $n = 1$ the theorem is obvious. We make the induction assumption that it is also true for R^{n-1}. The $(n-1)$-dimensional unit cube $W \cap \{(x_1, x_2, \ldots, x_n)\colon x_n = 0\}$ will be denoted by W', and the volume in R^{n-1} by v'. As a further notational convenience we set $c_n = 2 \cdot 3^{n-1} - 1$. For $i = 1, 2, \ldots, m$, let h_i be the maximum length of the edges of I_i, and let B_i be a corresponding base of I_i, that means an $(n-1)$-dimensional face of I_i which is orthogonal to an edge of I_i of length h_i. If there are several possibilities qualifying as B_i we assume that some selection has been made and is kept fixed. We shall also assume that every B_i is contained in the same plane as W', i.e. in $x_n = 0$. This can always be achieved by the performance of a rigid motion of I_i.

It is obvious that for every i,

$$h_i \leqslant d(I_i) \leqslant 1 \tag{5}$$

and

$$v(I_i) = h_i v'(B_i) \leqslant v'(B_i). \tag{6}$$

Furthermore, there is no loss in generality by assuming that

$$h_1 \geqslant h_2 \geqslant \cdots \geqslant h_m. \tag{7}$$

We define now inductively a finite sequence $k_0, k_1, \ldots, k_s$ of integers by the conditions that $k_0 = 0$ and that k_{j+1} is the smallest integer with

$$\sum_{i=k_j+1}^{k_{j+1}} v'(B_i) \geqslant c_{n-1}. \tag{8}$$

It follows that $0 = k_0 < k_1 < \cdots < k_s \leqslant m$, where it is always assumed that s is as large as possible. This assumption can be expressed by

$$\sum_{i=k_s+1}^{m} v'(B_i) < c_{n-1},$$

which implies (because of (6))

$$\sum_{i=k_s+1}^{m} v(I_i) < c_{n-1}. \tag{9}$$

It is also of importance to note that we are always concerned with the case $s \geqslant 1$ since (4) and (6) imply that

$$\sum_{i=1}^{m} v'(B_i) \geqslant \sum_{i=1}^{m} v(I_i) \geqslant c_n > c_{n-1}.$$

From the fact that k_{j+1} is the smallest number so that (8) holds it follows that

$$\sum_{i=k_j+1}^{k_{j+1}-1} v'(B_i) < c_{n-1},$$

and since (5) implies $v'(B_i) \leqslant d(B_i)^{n-1} \leqslant d(I_i)^{n-1} \leqslant 1$, we can infer that

$$\sum_{i=k_j+1}^{k_{j+1}} v'(B_i) < c_{n-1} + 1. \tag{10}$$

From (8) and the induction assumption it follows that for $j = 0, 1, \ldots, s - 1$ the cube W' can be covered by each of the collections $\{B_{k_j+1}, B_{k_j+2}, \ldots, B_{k_{j+1}}\}$. We may assume that these intervals are already arranged so that

$$W' \subset B_{k_j+1} \cup B_{k_j+2} \cup \cdots \cup B_{k_{j+1}}. \tag{11}$$

From (6), (7) and (10) we obtain now for $j = 0, 1, \ldots, s - 1$,

$$h_{k_j+1}(c_{n-1} + 1) \geqslant \sum_{i=k_j+1}^{k_{j+1}} h_{k_j+1} v'(B_i) \geqslant \sum_{i=k_j+1}^{k_{j+1}} h_i v'(B_i) = \sum_{i=k_j+1}^{k_{j+1}} v(I_i),$$

and therefore

$$\sum_{j=0}^{s-1} h_{k_j+1}(c_{n-1} + 1) \geqslant \sum_{j=0}^{s-1} \sum_{i=k_j+1}^{k_{j+1}} v(I_i) = \sum_{i=1}^{k_s} v(I_i)$$

$$= \sum_{i=1}^{m} v(I_i) - \sum_{i=k_s+1}^{m} v(I_i).$$

Because of (4) and (9) we can therefore deduce that

$$\sum_{j=0}^{s-1} h_{k_j+1} > \frac{c_n - c_{n-1}}{c_{n-1}+1} = 2.$$

As an immediate consequence of this inequality and (5) we find

$$\sum_{j=1}^{s-1} h_{k_j+1} > 2 - h_{k_0+1} \geqslant 1.$$

Since (7) shows that $h_{k_j+1} \leqslant h_{k_j}$ we obtain therefore

$$\sum_{j=1}^{s-1} h_{k_j} \geqslant \sum_{j=1}^{s-1} h_{k_j+1} > 1.$$

If in this inequality the term h_{k_s} is added to the left-hand side and j is replaced by $j + 1$, it follows that

$$\sum_{j=0}^{s-1} h_{k_{j+1}} > 1. \tag{12}$$

Because of (11) and (7), each union of the form $I_{k_j+1} \cup I_{k_j+2} \cup \cdots \cup I_{k_{j+1}}$ ($j = 0, 1, \ldots, s-1$) contains an interval J_j with W' as one face and an edge orthogonal to W' and of length $h_{k_{j+1}}$. Because of (12) it is clearly possible to translate the intervals J_j in the x_n-direction so that they cover W. This finishes the proof.

If $I_1, I_2, \ldots$ is an infinite sequence with $d(I_i) \leqslant 1$ and $\sum_{i=1}^{\infty} v(I_i) = \infty$, we have $v(I_i) \leqslant 1$, and it follows immediately from Theorem 2 that there is an integer m so that $I_i, I_2, \ldots, I_m$ cover W, but $\sum_{i=1}^{m} v(I_i) \leqslant 2 \cdot 3^{n-1}$. The following corollary is an obvious consequence of this latter inequality, together with (3) and the fact that a similarity transformation does not change the density of a covering (see also [3] for the definition of the covering density and the general properties used here).

COROLLARY 1. *Let* $\{K_i\}$ ($i = 1, 2, \ldots$) *be a class of compact convex sets of* R^n *with the property that* $\sum_{i=1}^{\infty} v(K_i) = \infty$ *and that the set of the diameters* $d(K_i)$ *is bounded. Then,* $\{K_i\}$ *can cover* R^n *with a density not greater than* $2 \cdot 3^{n-1} n^n$. *If all the sets* K_i *are intervals this upper bound for the density can be replaced by* $2 \cdot 3^{n-1}$.

The bound $2 \cdot 3^{n-1}$ for the densities of coverings by intervals is probably far from the best possible value. In fact, it appears to be unknown whether there exists a sequence of uniformly bounded intervals which permits only space coverings of density greater than 1. If the volumes of the intervals are bounded away from 0 one can even prove the following theorem.

THEOREM 3. *Let* $\{I_i\}$ ($i = 1, 2, \ldots$) *be a class of intervals in* R^n *with the property that there exist constants* M *and* c *such that for* $i = 1, 2, \ldots,$

$$d(I_i) \leqslant M \tag{13}$$

and

(14) $$0 < c \leqslant v(I_i).$$

Then $\{I_i\}$ *can cover* R^n *with density* 1.

PROOF. Let $s_i^1, s_i^2, \ldots, s_i^n$ denote the lengths of the edges of the interval I_i (in some arbitrary but fixed order). Since it follows from (13) that for every i and k,

$$s_i^k \leqslant d(I_i) \leqslant M,$$

and from (13) and (14) that

$$c \leqslant v(I_i) \leqslant s_i^k d(I_i)^{n-1} \leqslant s_i^k M^{n-1},$$

we obtain

$$0 < c/M^{n-1} \leqslant s_i^k \leqslant M.$$

Therefore, if we associate with each interval I_i the point $s_i = (s_i^1, s_i^2, \ldots, s_i^n)$ of R^n, it follows that these points belong to the compact subset $T = \{(x^1, x^2, \ldots, x^n): c/M^{n-1} \leqslant x^k \leqslant M\}$ of R^n. Consequently, there exists a subsequence of s_i that converges to a limit within T. An obvious consequence of this fact can be formulated in the following way: There exists a subsequence $J_1, J_2, \ldots$ of $I_1, I_2, \ldots$ with the property that the lengths of the edges, say $t_j^1, t_j^2, \ldots, t_j^n$ of J_j (taken in the previously assigned order) converge to positive limits. Hence, $\lim_{j\to\infty} t_j^k = t^k$ for some $t^k > 0$ and every $k = 1, 2, \ldots, n$. It is now convenient to assume that $t^1 = t^2 = \cdots = t^n = 1$. This assumption entails no real loss in generality since it can always be justified by the application of a suitable nonsingular affine transformation. Therefore, we can now proceed under the assumption that for $k = 1, 2, \ldots, n$,

(15) $$\lim_{j\to\infty} t_j^k = 1.$$

In order to find translations of the intervals $I_1, I_2, \ldots$ so that R^n is covered with density 1 we construct first a sequence of cubes $W_1, W_2, \ldots$ so that $R^n \subset \bigcup_{j=1}^{\infty} W_j$, int $W_p \cap$ int $W_q = \varnothing$ (for $p \neq q$) and that the edge lengths, say w_j, of the cubes W_j satisfy the conditions

(16) $$w_j < 1,$$

(17) $$\lim_{j\to\infty} w_j = 1.$$

It is not difficult to show that such sequences of cubes exist. For example, one can first subdivide R^n into the sets $S_1 = 1 \cdot 2W$, $S_2 = 2 \cdot 3W - 1 \cdot 2W$, $S_3 = 3 \cdot 4W - 2 \cdot 3W, \ldots$ (where W is again the unit cube, $\alpha W = \{\alpha x: x \in W\}$ and the minus sign denotes the set theoretic difference), and then it is easily established that each set S_m can be subdivided into $m + 1$ "layers" of cubes (surrounding $(m-1)mW$) of edge length $m/(m+1)$. The set of these cubes, ordered arbitrarily as a sequence, can obviously serve as the above sequence W_j.

Because of (15) and (16) it is possible to find for each W_j an interval J_{i_j} such that an appropriate translate, let us denote it by J_j', of J_{i_j} covers W_j.

Moreover, this selection can clearly be made so that no J_{i_j} is used more than once in this process. We have now to show that the sequence J_1', $J_2', \ldots$, which is certainly a covering of R^n, has density 1. For this purpose let B_r denote a solid sphere in R^n centered at the origin and of radius r, and let d_r be defined by

$$d_r = \frac{1}{v(B_r)} \sum_{J_j' \subset B_r} v(J_j').$$

Then the density δ of the covering $\{J_j'\}$ is given by $\delta = \lim_{r\to\infty} d_r$.

It follows from (15) and (17) that for any positive ε one can find a number N so that for all $j > N$, $v(J_j')/v(W_j) < 1 + \varepsilon$. Consequently, if we set $\sum_{j \leqslant N} v(J_j') = Q$ and notice that $J_j' \subset B_r$ implies $W_j \subset B_r$ we obtain

$$\begin{aligned} d_r &\leqslant \frac{Q}{v(B_r)} + \frac{1}{v(B_r)} \sum_{J_j' \subset B_r, j > N} v(J_j') \\ &\leqslant \frac{Q}{v(B_r)} + \frac{1}{v(B_r)} \sum_{W_j \subset B_r, j > N} (1+\varepsilon) v(W_j) \leqslant \frac{Q}{v(B_r)} + (1+\varepsilon). \end{aligned}$$

Since Q does not depend on r, we have $\lim_{r\to\infty} Q/v(B_r) = 0$ and, therefore,

$$\delta = \lim_{r\to\infty} d_r \leqslant 1 + \varepsilon.$$

If we note that for any covering $\delta \geqslant 1$ we obtain for $\varepsilon \to 0$ that $\delta = 1$. But at this point the proof of our theorem is not yet completely finished, since only a subclass of the originally given set of intervals has been used for the described covering. However, the intervals which have not been used, let us call them $K_1, K_2, \ldots$, can be distributed (by the application of translations) so thinly that they have density 0. For example, if each K_i is translated so that it has distance i^2 from the origin then, using (13) we find

$$\lim_{r\to\infty} \frac{1}{v(B_r)} \sum_{K_i \subset B_r} v(K_i) \leqslant \lim_{r\to\infty} \frac{1}{v(B_r)} \sum_{i^2 < r} M^n \leqslant \lim_{r\to\infty} \frac{\sqrt{r}}{v(B_r)} M^n = 0.$$

Since $\{J_j'\}$ has density 1 and $\{K_i\}$ has density 0 it follows that $\{J_j'\} \cup \{K_i\}$ has density 1, and the proof of Theorem 3 is finished.

The following corollary is an immediate consequence of (3) and Theorem 3. It is of the same type as Corollary 1 but it provides a better estimate under stronger assumptions.

COROLLARY 2. *Let* $\{K_i\}$ $(i = 1, 2, \ldots)$ *be a class of compact convex sets of* R^n *with the property that the set of the diameters* $d(K_i)$ *is bounded from above, and the set of the volumes* $v(K_i)$ *is bounded away from* 0. *Then* $\{K_i\}$ *can cover* R^n *with density not greater than* n^n.

Finally, it might be worth mentioning that Theorem 3 can also be formulated for packings instead of coverings and that the packing version can be proved by essentially the same methods as the covering theorem.

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