OSCILLATION OF A FORCED SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION

SAMUEL M. RANKIN III

Abstract. Sufficient conditions are given which insure that every solution of \((a(t)y')' + p(t)f(y)g(y') = r(t)\) has arbitrarily large zeros. We seem to have a partial answer to a question posed by Kartsatos [4]. An example is given illustrating the result.

Recently Keener [5], Skidmore and Bowers [8], Skidmore and Leighton [9], Rankin [7] and Tefteller [10] have investigated the oscillation and nonoscillation of solutions of the linear equations \((a(t)u')' + p(t)u(t) = r(t)\). In each of the above papers the results have depended on the associated homogeneous equation \((a(t)u')' + p(t)u(t) = 0\). Thus, these results are restrictive in the sense that the methods are not extendible to the nonlinear case.

Here we will study the equation

\[(a(t)y')' + p(t)f(y)g(y') = r(t).\]

Graef and Spikes [1] have recently given conditions which insure that every strictly positive solution or strictly negative solution or solutions that have arbitrarily large zeros but are of one sign approach zero as \(t\) approaches infinity. Their result extends a result of Hammett [2] for equation (1) when \(g(y') = 1\). Others who have investigated the behavior of solutions of (1) when \(g(y') = 1\) are Kartsatos [3], [4] and Londen [6]; Teufel [11] has investigated the case for \(p(t)f(y)g(y') = h(t, y, y')\).

Our result gives sufficient conditions which guarantee that every solution of equation (1) is oscillatory. As usual, a solution \(y(t)\) of (1) is oscillatory if for each real number \(a\), there exists a \(b > a\) such that \(y(b) = 0\). Obviously, we are assuming that all solutions can be extended to infinity.

The following assumptions are made: \(a(t), p(t),\) and \(r(t)\) are continuous on \([0, \infty)\); \(f(x)\) and \(g(x)\) are continuous on \((-\infty, \infty)\); \(a(t) > 0\) and \(p(t) > 0\) on \([0, \infty)\) and

(i) \(\int_0^\infty ds/a(s) = \infty;\)
(ii) \(\int_0^\infty p(s)ds = \infty;\)
(iii) \(xf(x) > 0\) for \(x \neq 0, df/dx \geq 0\) for all \(x;\)
(iv) \(g(x') \geq k > 0\) for all \(x';\)

Presented to the Society, January 22, 1976; received April 14, 1975 and, in revised form, July 25, 1975.

Key words and phrases. Nonoscillatory solution, oscillatory solution, oscillation, nonlinear forced differential equations.

© American Mathematical Society 1976
(v) \(|\int_{b}^{t} (\int_{s}^{t} dw/a(w))r(s)ds| < M\) for all \(t \geq b\) and all \(b > 0\).

(vi) There exists an increasing sequence \(\{b_n\}\) with \(\lim_{n \to \infty} b_n = \infty\) such that for each \(n\),

\[
\lim_{t \to \infty} \int_{b_n}^{t} \left( \int_{s}^{t} dw/a(w) \right) r(s) ds = L_n.
\]

The numbers \(L_n\) take on both positive, negative or possibly zero values for arbitrarily large \(n\).

**Theorem 1.** If conditions (iv)–(vi) are satisfied, then every solution \(y(t)\) of equation (1) is oscillatory.

**Proof.** First, we notice that for each \(n\),

\[
\int_{b_n}^{t} \left( \int_{s}^{t} dw/a(w) \right) r(s) ds
\]

\[
= \left( \int_{b_n}^{t} dw/a(w) \right) \int_{b_n}^{t} r(s) ds - \int_{b_n}^{t} \left( \int_{b_n}^{s} dw/a(w) \right) r(s) ds
\]

and so by (i), (v), and (vi),

\[
\lim_{n \to \infty} \int_{b_n}^{t} r(s) ds = 0 \quad \text{for all } n.
\]

Now assume that there exists a solution \(y(t)\) of equation (1) such that \(y(t) > 0\) on \([T_0, \infty)\) for some \(T_0 > 0\). We have by integration of (1) from \(b_n\) to \(t\) for some \(b_n > T_0\) that

\[
a(t)y'(t) + \int_{b_n}^{t} p(s)f(y)g(y')ds = a(b_n)y'(b_n) + \int_{b_n}^{t} r(s) ds.
\]

From (2) we have

\[
\lim_{t \to \infty} \left( a(t)y'(t) + \int_{b_n}^{t} p(s)f(y)g(y')ds \right) = a(b_n)y'(b_n).
\]

Since \(\int_{b_n}^{t} p(s)f(y)g(y')ds\) is nondecreasing with \(t\), we see that \(\lim_{t \to \infty} \int_{b_n}^{t} p(s)f(y)g(y')ds = \alpha\) where \(\alpha = \infty\) or \(\alpha\) is positive and finite. If \(\alpha = \infty\), then \(a(t)y'(t)\) must approach \(-\infty\) as \(t\) approaches \(\infty\) and this with (i) would imply that \(y(t)\) is eventually negative. Therefore, \(\alpha\) is finite.

Using (2) again we have that \(\lim_{t \to \infty} a(t)y'(t) = \beta\). The number \(\beta\) must be zero, for if \(\beta > 0\) then \(a(t)y'(t) \geq c > 0\) for all \(t \geq T_1 > T_0\) for some \(T_1\). From (i) it would then be true that \(\lim_{t \to \infty} y(t) = \infty\). Thus, by (iii) and (iv) there exists an \(L > 0\) such that \(f(y)g(y') > L\) for all \(t \geq T_2 > T_1\) for some \(T_2\). Integrating (1) from \(T_2\) to \(t\) we get

\[
a(t)y'(t) - a(T_2)y'(T_2) + L \int_{T_2}^{t} p(s) ds = \int_{T_2}^{t} r(s) ds.
\]

The right side of (4) is bounded as \(t\) goes to infinity while the left side goes to \(\infty\); hence, \(\beta\) is not positive.

If \(\beta < 0\), then there would exist an \(s > 0\) such that \(a(t)y'(t) < -s\) for all
OSCILLATION OF A NONLINEAR DIFFERENTIAL EQUATION

\( t \geq T_3 > T_0 \) for some \( T_3 \). This again leads to \( y(t) \) becoming negative. Therefore, we must have \( \lim_{t \to \infty} a(t)y'(t) = 0 \), and from (3),

\[
\lim_{t \to \infty} \int_{b_n}^{t} p(s)f(y)g(y')\,ds = a(b_n)y'(b_n). 
\]

From (5) and since \( p(t)f(y(t))q(y'(t)) > 0 \) for \( t \geq b_n \), we have that for each \( b_n \geq T_0 \)

\[
\int_{b_n}^{t} p(s)f(y)g(y')\,ds \leq a(b_n)y'(b_n) \quad \text{for all} \quad t \geq b_n.
\]

Assume that \( n \) is such that \( L_n > 0 \) and let the positive number \( \varepsilon \) be such that \( L_n - \varepsilon > 0 \) if \( L_n > 0 \) or \( y(b_n) - \varepsilon > 0 \) if \( L_n = 0 \). By integrating (1) twice from \( b_n \) to \( t \) we obtain

\[
y(t) = y(b_n) + \int_{b_n}^{t} \left( \int_{s}^{t} \frac{dw}{a(w)} \right) r(s)\,ds + a(b_n)y'(b_n) \int_{b_n}^{t} \frac{dw}{a(w)}
\]

\[- \int_{b_n}^{t} \left( \int_{s}^{t} \frac{dw}{a(w)} \right) p(s)f(y)g(y')\,ds \geq y(b_n) + L_n - \varepsilon
\]

\[+ \left( \int_{b_n}^{t} \frac{dw}{a(w)} \right) \left[ a(b_n)y'(b_n) - \int_{b_n}^{t} p(s)f(y)g(y')\,ds \right]
\]

\[\geq y(b_n) + L_n - \varepsilon = L' > 0 \quad \text{for all} \quad t \geq T_4 \text{ for some} \quad T_4 > T_0.
\]

Integrating (1) once more from \( T_4 \) to \( t \) we have

\[
a(t)y'(t) - a(T_4)y'(T_4) + f(L')k \int_{T_4}^{t} p(s)\,ds \leq \int_{T_4}^{t} r(s)\,ds.
\]

Since \( \lim_{t \to \infty} a(t)y'(t) = 0 \), we see from (ii) that the left side of (6) approaches \( \infty \) which contradicts the boundedness of \( \int_{T_4}^{t} r(s)\,ds \).

Hence, our assumption that \( y(t) > 0 \) on \([T_0, \infty)\) must be false (the case \( y(t) < 0 \) is similar and will, therefore, be omitted), implying that \( y(t) \) must be oscillatory.

**EXAMPLE.** Consider the equation

\[
y'' + p(t)f(y)g(y') = e^{-t} \sin t.
\]

We have that \( \left| \int_{b_n}^{t} (s - b)e^{-s} \sin s\,ds \right| \) is bounded for all \( b \) and all \( t \geq b \) and that

\[
\lim_{t \to \infty} \int_{b_n}^{t} (t - s)e^{-s} \sin s\,ds = L_n = \begin{cases} \sqrt{2}e^{-b_n/2} & \text{if } n \text{ even,} \\ -\sqrt{2}e^{-b_n/2} & \text{if } n \text{ odd,} \end{cases}
\]

where \( b_n = 3\pi/4 + n\pi, n = 0, 1, 2, \ldots \). Thus if conditions (ii)–(iv) are satisfied then every solution of (7) is oscillatory. We can see that condition (vi) is critical by considering the equation

\[
y'' + y = e^{-t}.
\]

Note that (i)–(v) are satisfied by (8) but (vi) is not. Equation (8) has the nonoscillatory solution \( y(t) = e^{-t/2} \).
Remarks. From our example, we see that our results handle functions \( r(t) \) which are not considered in [11]. Thus, along with the results of [11] we seem to have at least a partial answer to a question posed by Kartsatos [4].

We note that for the example considered, the distance between \( b_n \) and \( b_{n+1} \) remained constant for all \( n \) and that the \( L_n \)'s alternated between strictly positive and negative values. These facts, however, were not needed in the proof of the theorem. In fact, the case \( L_n = 0 \) for all \( n \) would be acceptable. If \( r(t) = 0 \), we would have this situation and therefore, it seems that our result is a natural extension of what is known for the equation
\[
(a(t)u')' + p(t)f(u)g(u') = 0.
\]

Bibliography


Department of Mathematics, Murray State University, Murray, Kentucky 42701