

A REMARK ON STRONGLY EXPOSING FUNCTIONALS

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ABSTRACT. By using the concept of farthest points, we show that the set of strongly exposing functionals of a weakly compact convex subset in a Banach space X is a dense G_δ in X^* . The construction also gives a new proof of existence of strongly exposed points in weakly compact convex sets.

Let K be a convex subset in a Banach space X , a point $x \in K$ is called a *strongly exposed point* of K if there exists an $f \in X^*$ such that (i) $f(x) > f(y)$ for all $y \neq x$ in K , (ii) for any sequence (x_n) in K with $f(x_n) \rightarrow f(x)$, $x_n \rightarrow x$ in norm. We call the above f a *strongly exposing functional* of K and use K^Δ to denote the set of strongly exposing functionals of K . Lindenstrauss [5] and Troyanski [6] proved that if K is a weakly compact convex subset in X , then K is the closed convex hull of its strongly exposed points. In [1], Anantharaman showed that if K is the closed convex hull of the range of a vector-valued measure (hence K is weakly compact) then K^Δ is a dense G_δ in X^* . A similar conclusion has also been obtained by the author for weakly compact convex subsets in certain classes of Banach spaces [4]. In this note, by modifying the method in [4], we prove

THEOREM 1. *Let K be a weakly compact convex subset in a Banach space X ; then K^Δ is a dense G_δ in X^* .*

In the proof, we will need the following propositions.

PROPOSITION 2 (TROYANSKI). *Let X be a weakly compact generated Banach space; then X admits an equivalent locally uniformly convex norm.*

PROPOSITION 3 (LAU). *Let K be a weakly compact subset in a Banach space X ; then the set*

$$\{x \in X: \|x - z\| = \sup\{\|x - y\|: y \in K\} \text{ for some } z \in K\}$$

is a dense G_δ in X .

We call the point z in the above proposition a *farthest point* of K [2], [3]. It is known that if X is locally uniformly convex, then a farthest point of a

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bounded convex subset is also a strongly exposed point.

PROOF OF THE THEOREM. Note that

$$K^\Lambda = \bigcap_{n=1}^\infty \left\{ f \in X^* : \text{diam} \left\{ x \in K : f(x) > \sup_{y \in K} f(y) - a \right\} < \frac{1}{n} \text{ for some } a > 0 \right\}$$

and the set on the right side is a G_δ [1], [4]; hence it suffices to show the density of K^Λ in X^* . By a remark in [4] and Proposition 2, we may assume that X is weakly compact generated (say, by K) and locally uniformly convex. Let $f \in X^*$ with $\|f\| = 1$. For $\epsilon > 0$, let $C = f^{-1}(0) \cap 2\epsilon^{-1}B$ where B is the closed unit ball of X . By a homothetic translation, we may let $K \subseteq B$ but $K \not\subseteq C$ (note that K^Λ is unchanged). We will construct a point $z \in K$ which is a strongly exposed point of the closed set $\text{conv}(K \cup C)$. The corresponding strongly exposing functional g of $\text{conv}(K \cup C)$ with $\|g\| = 1$ will satisfy $\|f - g\| \leq \epsilon$ and also strongly exposes K at z (for details, cf. [4, Theorem 2.4]); hence this completes the proof.

Choose a point $x_1 \in K \setminus C$ such that the set

$$S = \{ \alpha x_1 + \beta y : |\alpha|^2 + |\beta|^2 \leq 1, y \in C \}$$

does not contain K (we neglect the case that K is a singleton, x_1 may be chosen as midpoint of some line segment of K not lying in C). Since C is an absorbent subset of the hyperplane $f^{-1}(0)$, S is an absorbent subset of X . Let $\|\cdot\|$ be the norm defined by S ; then $\|\cdot\|$ is locally uniformly convex and equivalent to the original norm. There exists $x_2 \in K \setminus S$ with $\|x_2\| - 1 = \alpha > 0$. By Proposition 3, there exist points $w \in X, z \in K$ with $\|w\| \leq \alpha/2$ and $\beta = \|w - z\| = \sup\{\|w - y\| : y \in K\}$. For any point $y \in C$,

$$\|y - w\| \leq \|y\| + \alpha/2 \leq 1 + \alpha/2 < \beta.$$

Hence z is also a farthest point of $\text{conv}(K \cup C)$. It follows that z is a strongly exposed point of $\text{conv}(K \cup C)$. Q.E.D.

We remark that the above construction yields another proof of the existence of strongly exposed points in weakly compact sets as in [5]. Moreover, we have

COROLLARY 4. *Let K be a weakly compact convex subset in a Banach space X ; then for any bounded closed convex subset C such that $K \not\subseteq C$, there exists a point $x \in K$ which strongly exposes $\text{conv}(K \cup C)$.*

PROOF. It follows easily from the above theorem and Theorem 2.4 in [4]: if K is a bounded closed convex subset in X , then K^Λ is a dense G_δ if and only if for any bounded closed convex subset C such that $K \not\subseteq C$, there exists a point $x \in K$ which is a strongly exposed point of $\text{conv}(K \cup C)$.

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