A REMARK ON STRONGLY EXPOSING FUNCTIONALS

KA-SING LAU

Abstract. By using the concept of farthest points, we show that the set of strongly exposing functionals of a weakly compact convex subset in a Banach space $X$ is a dense $G_δ$ in $X^*$. The construction also gives a new proof of existence of strongly exposed points in weakly compact convex sets.

Let $K$ be a convex subset in a Banach space $X$, a point $x \in K$ is called a strongly exposed point of $K$ if there exists an $f \in X^*$ such that (i) $f(x) > f(y)$ for all $y \neq x$ in $K$, (ii) for any sequence $(x_n)$ in $K$ with $f(x_n) \to f(x)$, $x_n \to x$ in norm. We call the above $f$ a strongly exposing functional of $K$ and use $K^λ$ to denote the set of strongly exposing functionals of $K$. Lindenstrauss [5] and Troyanski [6] proved that if $K$ is a weakly compact convex subset in $X$, then $K$ is the closed convex hull of its strongly exposed points. In [1], Anantharaman showed that if $K$ is the closed convex hull of the range of a vector-valued measure (hence $K$ is weakly compact) then $K^λ$ is a dense $G_δ$ in $X^*$. A similar conclusion has also been obtained by the author for weakly compact convex subsets in certain classes of Banach spaces [4]. In this note, by modifying the method in [4], we prove

**Theorem 1.** Let $K$ be a weakly compact convex subset in a Banach space $X$; then $K^λ$ is a dense $G_δ$ in $X^*$.

In the proof, we will need the following propositions.

**Proposition 2 (Troyanski).** Let $X$ be a weakly compact generated Banach space; then $X$ admits an equivalent locally uniformly convex norm.

**Proposition 3 (Lau).** Let $K$ be a weakly compact subset in a Banach space $X$; then the set

$$\{x \in X: \|x - z\| = \sup\{\|x - y\|: y \in K\} \text{ for some } z \in K\}$$

is a dense $G_δ$ in $X$.

We call the point $z$ in the above proposition a farthest point of $K$ [2], [3]. It is known that if $X$ is locally uniformly convex, then a farthest point of a...
bounded convex subset is also a strongly exposed point.

**Proof of the theorem.** Note that

\[ K^\lambda = \bigcap_{n=1}^{\infty} \left\{ f \in X^*: \text{diam} \left\{ x \in K : f(x) > \sup_{y \in K} f(y) - a \right\} < \frac{1}{n} \text{ for some } a > 0 \right\} \]

and the set on the right side is a \( G_\delta \) [1], [4]; hence it suffices to show the density of \( K^\lambda \) in \( X^* \). By a remark in [4] and Proposition 2, we may assume that \( X \) is weakly compact generated (say, by \( K \)) and locally uniformly convex. Let \( f \in X^* \) with \( \|f\| = 1 \). For \( \epsilon > 0 \), let \( C = f^{-1}(0) \cap 2\epsilon^{-1}B \) where \( B \) is the closed unit ball of \( X \). By a homothetic translation, we may let \( K \subseteq B \) but \( K \not\subseteq C \) (note that \( K^\lambda \) is unchanged). We will construct a point \( z \in K \) which is a strongly exposed point of the closed set \( \text{conv}(K \cup C) \). The corresponding strongly exposing functional \( g \) of \( \text{conv}(K \cup C) \) with \( \|g\| = 1 \) will satisfy \( \|f - g\| \leq \epsilon \) and also strongly exposes \( K \) at \( z \) (for details, cf. [4, Theorem 2.4]); hence this completes the proof.

Choose a point \( x_1 \in K \setminus C \) such that the set

\[ S = \{ \alpha x_1 + \beta y : |\alpha|^2 + |\beta|^2 < 1, y \in C \} \]

does not contain \( K \) (we neglect the case that \( K \) is a singleton, \( x_1 \) may be chosen as midpoint of some line segment of \( K \) not lying in \( C \)). Since \( C \) is an absorbent subset of the hyperplane \( f^{-1}(0) \), \( S \) is an absorbent subset of \( X \). Let \( \|\cdot\| \) be the norm defined by \( S \); then \( |||\cdot||| \) is locally uniformly convex and equivalent to the original norm. There exists \( x_2 \in K \setminus S \) with \( ||x_2|| - 1 = \alpha > 0 \). By Proposition 3, there exist points \( w \in X, z \in K \) with \( \|w\| \leq \alpha/2 \) and \( \beta = \|w - z\| = \sup\{\|w - y\| : y \in K\} \). For any point \( y \in C \),

\[ \|y - w\| \leq \|w\| + \alpha/2 \leq 1 + \alpha/2 \leq \beta. \]

Hence \( z \) is also a farthest point of \( \text{conv}(K \cup C) \). It follows that \( z \) is a strongly exposed point of \( \text{conv}(K \cup C) \). Q.E.D.

We remark that the above construction yields another proof of the existence of strongly exposed points in weakly compact sets as in [5]. Moreover, we have

**Corollary 4.** Let \( K \) be a weakly compact convex subset in a Banach space \( X \); then for any bounded closed convex subset \( C \) such that \( K \not\subseteq C \), there exists a point \( x \in K \) which strongly exposes \( \text{conv}(K \cup C) \).

**Proof.** It follows easily from the above theorem and Theorem 2.4 in [4]: if \( K \) is a bounded closed convex subset in \( X \), then \( K^\lambda \) is a dense \( G_\delta \) if and only if for any bounded closed convex subset \( C \) such that \( K \not\subseteq C \), there exists a point \( x \in K \) which is a strongly exposed point of \( \text{conv}(K \cup C) \).
REFERENCES


Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260