PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, September 1976

A REMARK ON STRONGLY EXPOSING FUNCTIONALS

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ABSTRACT. By using the concept of farthest points, we show that the set of strongly exposing functionals of a weakly compact convex subset in a Banach space X is a dense G_δ in X^*. The construction also gives a new proof of existence of strongly exposed points in weakly compact convex sets.

Let K be a convex subset in a Banach space X, a point $x \in K$ is called a *strongly exposed point* of K if there exists an $f \in X^*$ such that (i) $f(x) > f(y)$ for all $y \neq x$ in K, (ii) for any sequence (x_n) in K with $f(x_n) \to f(x)$, $x_n \to x$ in norm. We call the above f a *strongly exposing functional* of K and use K^Λ to denote the set of strongly exposing functionals of K. Lindenstrauss [5] and Troyanski [6] proved that if K is a weakly compact convex subset in X, then K is the closed convex hull of its strongly exposed points. In [1], Anantharaman showed that if K is the closed convex hull of the range of a vector-valued measure (hence K is weakly compact) then K^Λ is a dense G_δ in X^*. A similar conclusion has also been obtained by the author for weakly compact convex subsets in certain classes of Banach spaces [4]. In this note, by modifying the method in [4], we prove

THEOREM 1. *Let K be a weakly compact convex subset in a Banach space X; then K^Λ is a dense G_δ in X^*.*

In the proof, we will need the following propositions.

PROPOSITION 2 (TROYANSKI). *Let X be a weakly compact generated Banach space; then X admits an equivalent locally uniformly convex norm.*

PROPOSITION 3 (LAU). *Let K be a weakly compact subset in a Banach space X; then the set*

$$\{x \in X\colon \|x - z\| = \sup\{\|x - y\|\colon y \in K\} \text{ for some } z \in K\}$$

is a dense G_δ in X.

We call the point z in the above proposition a *farthest point* of K [2], [3]. It is known that if X is locally uniformly convex, then a farthest point of a

Received by the editors September 16, 1975.
AMS (MOS) subject classifications (1970). Primary 46B99.
Key words and phrases. Banach spaces, farthest points, locally uniformly convex, strongly exposed points, strongly exposing functionals.

bounded convex subset is also a strongly exposed point.

PROOF OF THE THEOREM. Note that

$$K^{\Lambda} = \bigcap_{n=1}^{\infty} \left\{ f \in X^*: \operatorname{diam} \left\{ x \in K: f(x) > \sup_{y \in K} f(y) - a \right\} < \frac{1}{n} \text{ for some } a > 0 \right\}$$

and the set on the right side is a G_δ [1], [4]; hence it suffices to show the density of K^{Λ} in X^*. By a remark in [4] and Proposition 2, we may assume that X is weakly compact generated (say, by K) and locally uniformly convex. Let $f \in X^*$ with $\|f\| = 1$. For $\varepsilon^{>0}$, let $C = f^{-1}(0) \cap 2\varepsilon^{-1}B$ where B is the closed unit ball of X. By a homothetic translation, we may let $K \subseteq B$ but $K \not\subseteq C$ (note that K^{Λ} is unchanged). We will construct a point $z \in K$ which is a strongly exposed point of the closed set conv $(K \cup C)$. The corresponding strongly exposing functional g of conv $(K \cup C)$ with $\|g\| = 1$ will satisfy $\|f - g\| \leqslant \varepsilon$ and also strongly exposes K at z (for details, cf. [4, Theorem 2.4]); hence this completes the proof.

Choose a point $x_1 \in K \backslash C$ such that the set

$$S = \{\alpha x_1 + \beta y: |\alpha|^2 + |\beta|^2 \leqslant 1, y \in C\}$$

does not contain K (we neglect the case that K is a singleton, x_1 may be chosen as midpoint of some line segment of K not lying in C). Since C is an absorbent subset of the hyperplane $f^{-1}(0)$, S is an absorbent subset of X. Let $|||\cdot|||$ be the norm defined by S; then $|||\cdot|||$ is locally uniformly convex and equivalent to the original norm. There exists $x_2 \in K\backslash S$ with $|||x_2||| - 1 = \alpha > 0$. By Proposition 3, there exist points $w \in X$, $z \in K$ with $|||w||| \leqslant \alpha/2$ and $\beta = |||w - z||| = \sup\{|||w - y|||: y \in K\}$. For any point $y \in C$,

$$|||y - w||| \leqslant |||y||| + \alpha/2 \leqslant 1 + \alpha/2 < \beta.$$

Hence z is also a farthest point of conv $(K \cup C)$. It follows that z is a strongly exposed point of conv $(K \cup C)$. Q.E.D.

We remark that the above construction yields another proof of the existence of strongly exposed points in weakly compact sets as in [5]. Moreover, we have

COROLLARY 4. *Let K be a weakly compact convex subset in a Banach space X; then for any bounded closed convex subset C such that $K \not\subseteq C$, there exists a point $x \in K$ which strongly exposes* conv $(K \cup C)$.

PROOF. It follows easily from the above theorem and Theorem 2.4 in [4]: if K is a bounded closed convex subset in X, then K^{Λ} is a dense G_δ if and only if for any bounded closed convex subset C such that $K \not\subseteq C$, there exists a point $x \in K$ which is a strongly exposed point of $\overline{\text{conv}}\,(K \cup C)$.

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